

# Efficient Key Recovery for all HFE Signature Variants

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# Multivariate Cryptography

Public Key: System of multivariate quadratic polynomials

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Security based on the

**Problem MQ:** Given  $m$  multivariate quadratic polynomials  $p^{(1)}(\mathbf{x}), \dots, p^{(m)}(\mathbf{x})$ , find a vector  $\bar{\mathbf{x}} = (\bar{x}_1, \dots, \bar{x}_n)$  such that  $p^{(1)}(\bar{\mathbf{x}}) = \dots = p^{(m)}(\bar{\mathbf{x}}) = 0$ .

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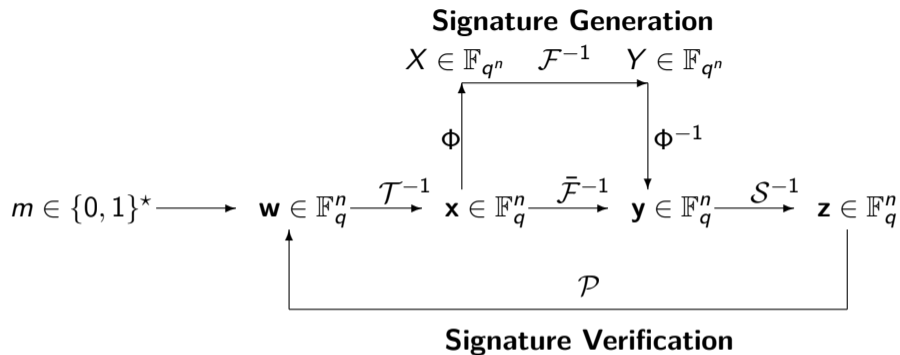
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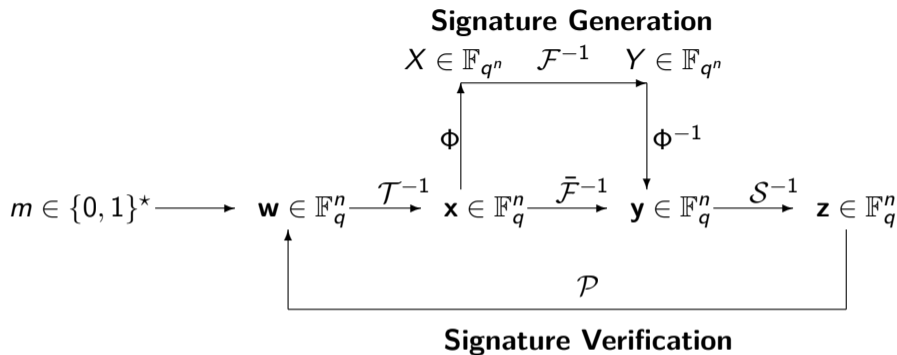
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- *Private key*:  $\mathcal{T}, \mathcal{F}, \mathcal{S}$  allows to invert the public key



# BigField Signature Schemes

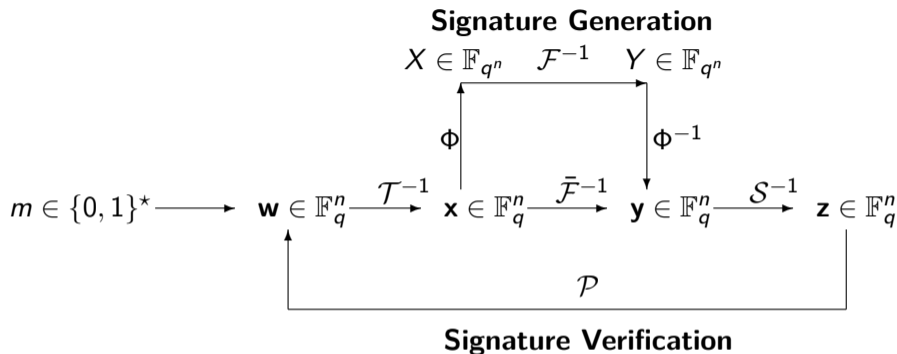


# BigField Signature Schemes



**Signature Generation:** Given: message  $m \in \{0, 1\}^*$ , private key  $\mathcal{T}, \mathcal{F}, \mathcal{S}$   
compute recursively  $\mathbf{w} = \mathcal{H}(m)$ ,  $\mathbf{x} = \mathcal{T}^{-1}(\mathbf{w})$ ,  $X = \Phi(\mathbf{x})$ ,  $Y = \mathcal{F}^{-1}(X)$ ,  $\mathbf{y} = \Phi^{-1}(Y)$  and  
 $\mathbf{z} = \mathcal{S}^{-1}(\mathbf{y})$

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**Signature Verification:** Given: message  $m \in \{0, 1\}^*$ , signature  $\mathbf{z} \in \mathbb{F}_q^n$ , public key  $\mathcal{P}$   
 check if  $\mathcal{P}(\mathbf{z}) = \mathcal{H}(m)$

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Signature:  $\mathbf{z} \in \mathbb{F}_q^{n+v}$ .

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- Accept the signature  $\mathbf{z} \Leftrightarrow \mathbf{w}' = \mathbf{w}$ .

## Previous Attacks against HFEv<sup>-</sup>

- direct attack (signature forgery)  
degree of regularity is bounded by

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# Preliminaries

- We use the matrix representation of the HFE central map, i.e.

$\mathcal{F}(X, x_1, \dots, x_v) = (X, X^q, \dots, X^{q^{n-1}}, x_1, \dots, x_v) F^{*0} (X, X^q, \dots, X^{q^{n-1}}, x_1, \dots, x_v)^t$  with

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$\Rightarrow$  We get

$$\mathcal{F}^{q^k}(X, x_1, \dots, x_v) = (X, X^q, \dots, X^{q^{n-1}}, x_1, \dots, x_v) F^{*k} (X, X^q, \dots, X^{q^{n-1}}, x_1, \dots, x_v)^t$$

## Preliminaries (2)

- We use a morphism  $\Phi : \mathbb{F}_{q^n} \rightarrow \mathbb{F}_q^n$  given by the matrix

$$M = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \theta & \theta^q & \dots & \theta^{q^{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \theta^{n-1} & (\theta^{n-1})^q & \dots & (\theta^{n-1})^{q^{n-1}} \end{pmatrix}, \quad (1)$$

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- In order to cover the Vinegar variables, we define

$$\widetilde{M} = \begin{pmatrix} M & 0 \\ 0 & I_v \end{pmatrix}$$

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- Let  $S$  and  $T$  be the matrices representing the linear parts of  $\mathcal{S}$  and  $\mathcal{T}$ . From

$$\mathcal{F} = (X, X^q, \dots, X^{q^{n-1}}, x_1, \dots, x_v) F^{*0} (X, X^q, \dots, X^{q^{n-1}}, x_1, \dots, x_v)^t,$$

we find

$$\begin{aligned} & \left( \widetilde{M}^{-1} S^{-1} P_0 (S^{-1})^t (\widetilde{M}^{-1})^t, \dots, \widetilde{M}^{-1} S^{-1} P_{n-a-1} (S^{-1})^t (\widetilde{M}^{-1})^t \right) \\ &= \left( F^{*0}, \dots, F^{*(n-1)} \right) M^{-1} T \end{aligned}$$

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$$(v_1, v_2, \dots, v_n, x_1, \dots, x_v) \cdot \widetilde{M} = (V, V^q, \dots, V^{q^{n-1}}, x_1, \dots, x_v),$$

- Let  $S$  and  $T$  be the matrices representing the linear parts of  $\mathcal{S}$  and  $\mathcal{T}$ . From

$$\mathcal{F} = (X, X^q, \dots, X^{q^{n-1}}, x_1, \dots, x_v) F^{*0} (X, X^q, \dots, X^{q^{n-1}}, x_1, \dots, x_v)^t,$$

we find

$$\begin{aligned} & \left( \widetilde{M}^{-1} S^{-1} P_0 (S^{-1})^t (\widetilde{M}^{-1})^t, \dots, \widetilde{M}^{-1} S^{-1} P_{n-a-1} (S^{-1})^t (\widetilde{M}^{-1})^t \right) \\ &= \left( F^{*0}, \dots, F^{*(n-1)} \right) M^{-1} T \end{aligned}$$

- Denoting  $U = \widetilde{M}^{-1} S^{-1}$  and  $W = M^{-1} T$  yields

$$(UP_0 U^t, \dots, UP_{n-a-1} U^t) = (F^{*0}, \dots, F^{*(n-1)}) W.$$

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## Lemma

The rank of the matrix  $Q = W^t \cdot \begin{pmatrix} \mathbf{a}_0 \\ \vdots \\ \mathbf{a}_{n-1} \end{pmatrix}$  is at most  $d = \lceil \log_q(D) \rceil$ .

In particular, we have

$$\begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \dots \\ \mathbf{a}_{n-1} \end{pmatrix} = \begin{pmatrix} A_1 \\ 0 \\ A_2 \end{pmatrix},$$

## Recovering $S$ (2)

- From the Lemma we directly follow

### Theorem

Let  $\mathbf{u} = (u_0, u_1, \dots, u_{n+v-1})$  be the first row of  $U$  and  $\mathbf{b}_i = (u_0, u_1, \dots, u_{n+v-1})P_i$ , ( $i = 0, 1, \dots, n-a$ ). Define  $Z \in \mathcal{M}_{(n-a) \times (n+v)}(\mathbb{F}_{q^n})$  as the matrix whose row vectors are the  $\mathbf{b}_i$ . Then the rank of  $Z$  is at most  $d$ .

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- Furthermore we get

### Lemma

Let  $A$  be an  $m \times n$  matrix over  $\mathbb{F}_q$  and  $B = [b_{ij}] = M^{-1}A$ . Then we have

$$b_{ij} = b_{i-1,j}^q, \quad \text{for all } i, j, \text{ with } 0 \leq i < n, 0 \leq j < m.$$

i.e. the matrix  $B$  is determined by its first row.

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- We can find the first  $n$  of  $U$  using the Frobenius isomorphism
- The remaining rows of  $U$  can be chosen at random such that  $U$  is invertible

## Recovering $S$ - The Algorithm

**Input:** HFEv- parameters  $(q, n, v, D, a)$ , matrices  $(P_0, \dots, P_{n-a-1})$ , matrix  $\widetilde{M}$

**Output:** Equivalent linear transformation  $S$ .

- 1: Set  $\mathbf{b}_i = (1, u_1, \dots, u_{n+v-1})P_i$ ,  $0 \leq i < n - a$ , where  $(u_1, \dots, u_{n+v-1})$  are unknowns.
- 2: Construct a matrix  $Z$  whose row vectors are  $\mathbf{b}_i$ ,  $0 \leq i < n - a$ .
- 3: Solve a MinRank problem for  $Z$  to find the unknowns  $u_1, \dots, u_{n+v-1}$ .

4: Set  $U = \begin{pmatrix} 1 & u_1 & \cdots & u_{n+v-1} \\ 1 & u_1^q & \cdots & u_{n+v-1}^q \\ \vdots & \vdots & \ddots & \vdots \\ 1 & u_1^{q^{n-1}} & \cdots & u_{n+v-1}^{q^{n-1}} \\ r_{00} & r_{01} & \cdots & r_{0,n+v-1} \\ \vdots & \vdots & \ddots & \vdots \\ r_{v-1,0} & r_{v-1,1} & \cdots & r_{v-1,n+v-1} \end{pmatrix}$ ,

5: Compute  $S' = (\widetilde{M}U)^{-1}$ .

6: **return**  $S'$ .

# Recovering $\mathcal{F}$ and $\mathcal{T}$

- We can show

## Lemma

*As soon as  $U$  is known, we can recover  $F^{*0}$  by solving a determined linear system with  $n - a - 1$  variables,  $(d + a) \cdot (n + v)$  additional linear equations in at most  $d + v$  variables, and  $\binom{v+1}{2}$  univariate polynomial equations of degree  $q^d$ .*



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- Furthermore we get

### Lemma

*As soon as the matrices  $F^{*j}$  ( $0 \leq j < n$ ) are known,  $T$  can be recovered by solving a system of  $n - a$  linear equations in  $n$  variables.*

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- Support Minors Modelling  
we don't have a unique solution of the MinRank Problem  $\Rightarrow$  We solve the system by  $F_4$   
Experiments  $\Rightarrow$  degree of regularity 3

$$\mathcal{O} \left( (n + v)^2 \binom{2d + 2}{d} + (n + v) \binom{2d + 2}{d}^2 \right)^\omega$$

# Application to GeMSS

NIST security category		parameters ( $q, n, v, D, a$ )	required security level	our attack using	
				minors modeling	support minors modeling
I	GeMSS128	(2,174,12,513,12)	143	139	118
	BlueGeMSS128	(2,175,14,129,13)		119	99
	RedGeMSS128	(2,177,15,17,15)		86	72
II	GeMSS192	(2,265,20,513,22)	207	154	120
	BlueGeMSS192	(2,265,23,129,22)		132	101
	RedGeMSS192	(2,266,25,17,23)		95	75
III	GeMSS256	(2,354,33,513,30)	272	166	121
	BlueGeMSS256	(2,358,32,129,34)		141	103
	RedGeMSS256	(2,358,35,17,34)		101	76

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**The Techniques used in GeMSS don't suffice to create a HFE based signature scheme which is both efficient and reaches high levels of security**

## Conclusion

We proposed a new MinRank type attack against HFE signature variants. The complexity is

$$\mathcal{O} \left( \binom{n+d+v+1}{d+1}^\omega \right)$$

- exponential in  $d$
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- We can't speed up  $\text{HFEv}^-$  by decreasing  $d$  while increasing  $a$  and  $v$
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- For high levels of security we need a large  $d$

⇒ Can we build an HFE based signature scheme which is both efficient and offers a high level of security?

Thank you for your Attention

Find our paper at  
<https://eprint.iacr.org/2020/1424.pdf>