A Simple and Provably Good Code for SHA Message Expansion

Charanjit S. Jutla
IBM T. J. Watson

Anindya C. Patthak
Univ. of Texas at Austin

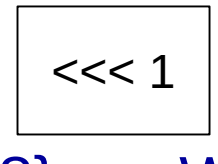
Key (Message) Expansion

- MD5: Simple bit permutations
- SHA0: Linear Code (LFSR style)
- SHA1: Linear Code (LFSR with rotations)
  - Not Good Enough (wt 25 in last 60 words)

SHA-1 Code

SHA-1:

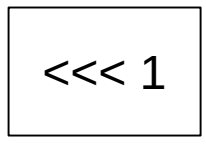
- W_i = (W_{i-3} xor W_{i-8} xor W_{i-14} xor W_{i-16}) >>>1



SHA-1 Backwards:

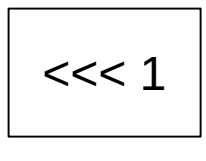
W_i <<<1 = W_{i-3} xor W_{i-8} xor W_{i-14} xor W_{i-16}

Or



W_{i-16} = (W_i <<< 1) xor W_{i-3} xor W_{i-8} xor W_{i-14}

Or



W_i = (W_{i+16}<<<1) xor W_{i+13} xor W_{i+8} xor W_{i+2}

Improved SHA-1 Code.0

- W_i = (W_{i-3} xor W_{i-8} xor W_{1-16})<<< 1
  xor
  W_{i-14}

Not easy to prove good lower bound (if any)
on this code

Improved SHA-1 Code.1

- W_i = W_{i-3} xor W_{i-8} xor W_{i-14} xor W_{i-16}
  xor
  (W_{i-1} xor W_{i-2} xor W_{i-15}) >>>1

- How do you prove a lower bound?
- Huge Dimension : 32*16

- Computer Assisted Proof

Novel Technique

(1) Either all columns are non-zero

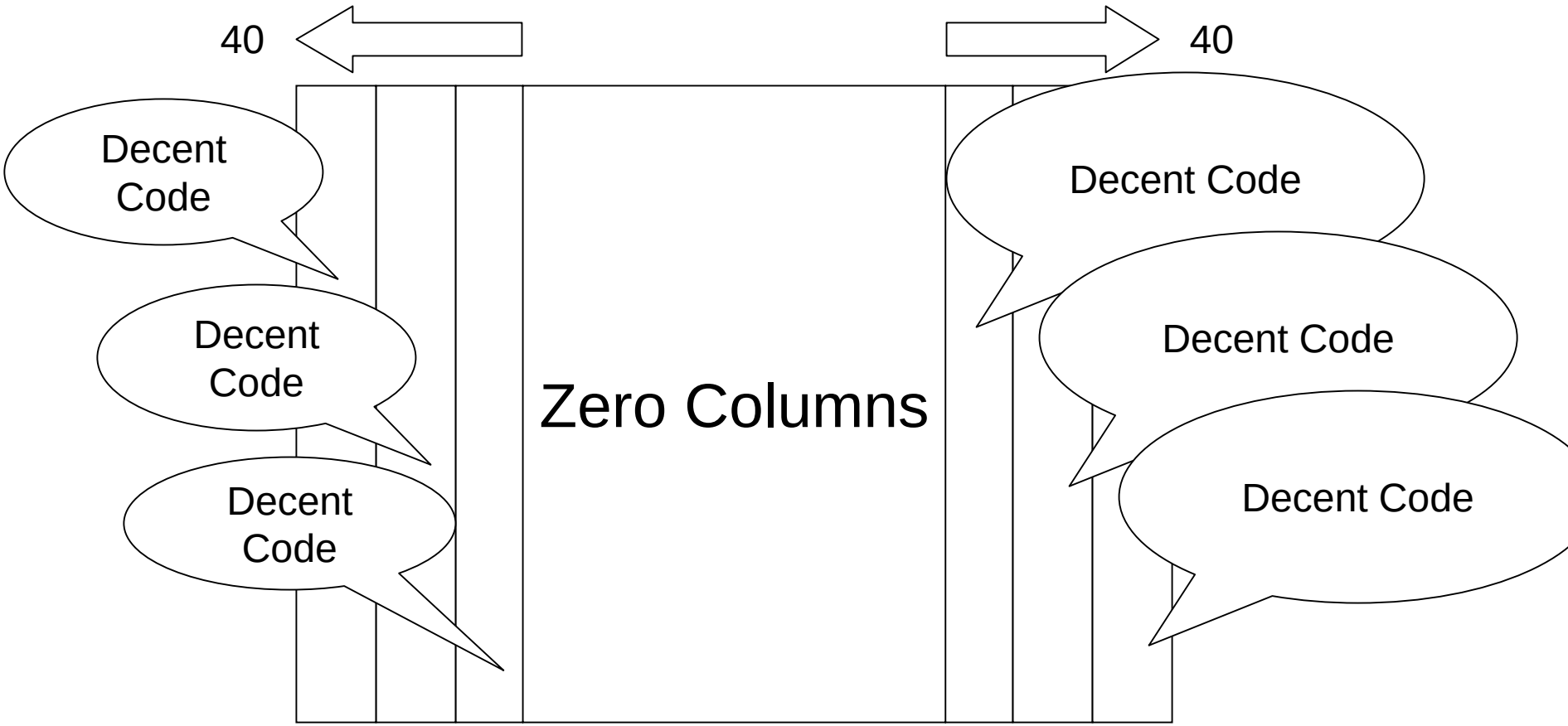
- prove average 3 bits ON per column

(2) Or some column is zero, and another column is non-zero

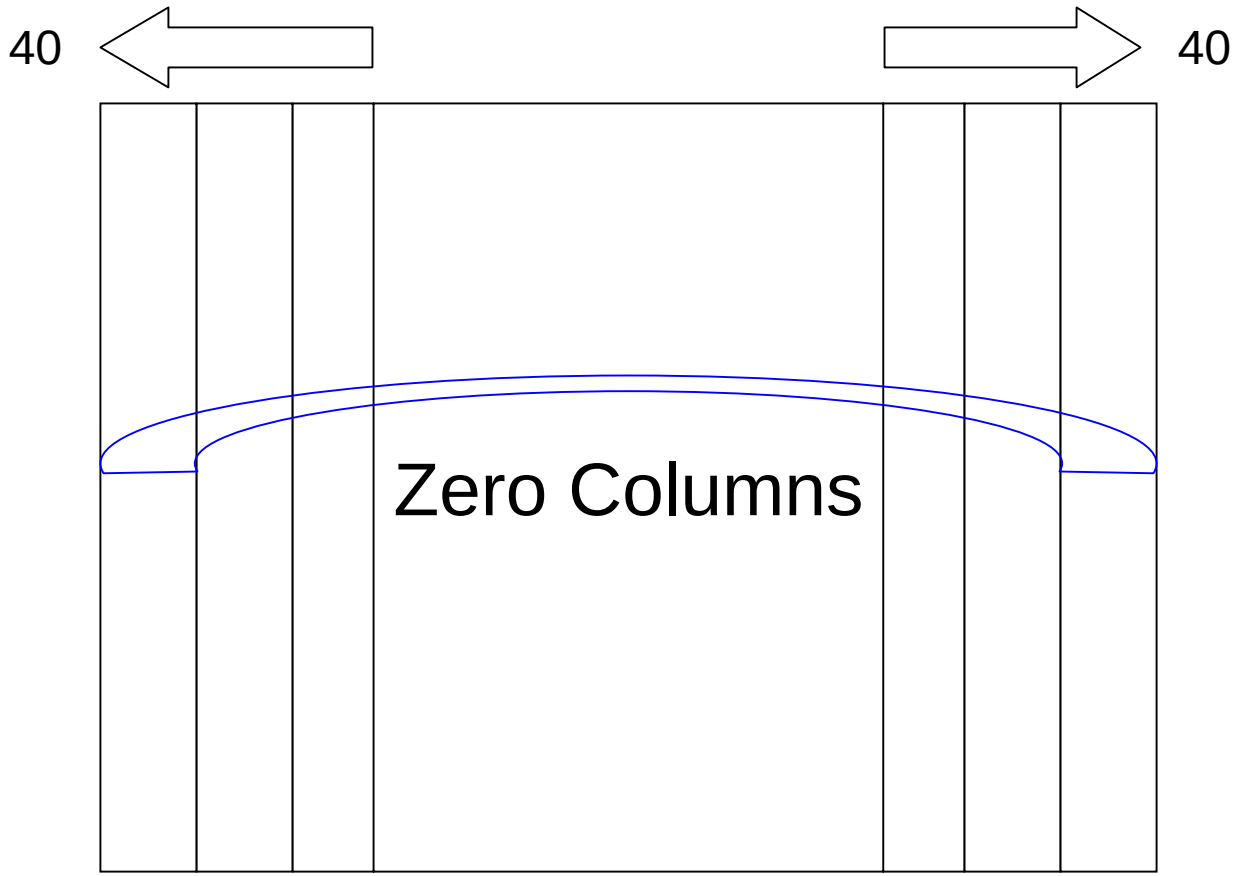
- good code, except for …

(3) Pathological Cases of low dimension

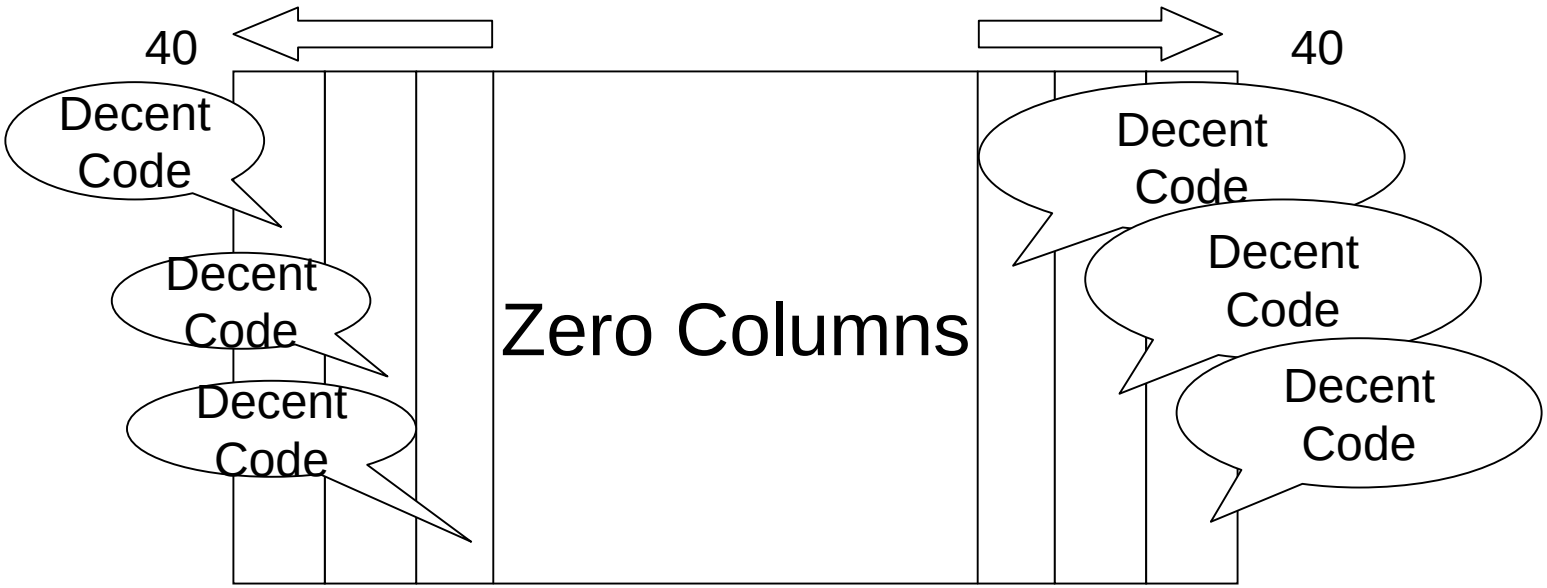
Band of Zero columns



Over Counting?

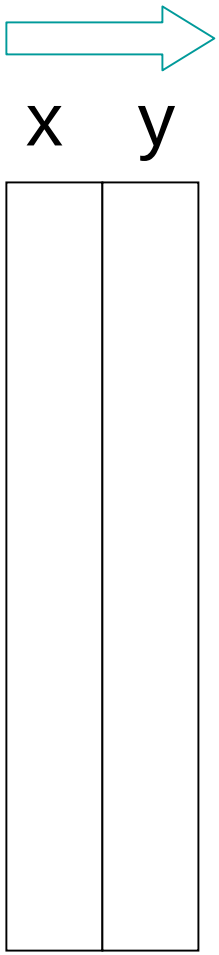


Band of Zero columns



$$W_i = W_{i-3} \text{ xor } W_{i-8} \text{ xor } W_{i-14} \text{ xor } W_{i-16} \text{ xor } (W_{i-1} \text{ xor } W_{i-2} \text{ xor } W_{i-15}) >>>1$$

W_i = W_{i-3} xor W_{i-8} xor W_{i-14} xor W_{i-16}
xor
(W_{i-1} xor W_{i-2} xor W_{i-15}) >>>1



If x=0 then

y_i = y_{i-3} xor y_{i-8}
xor y_{i-14} xor y_{i-16}

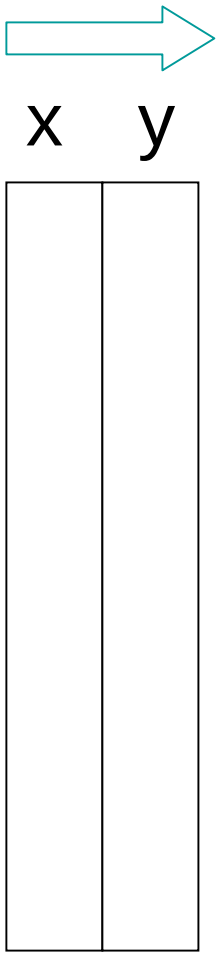
--- decent code (dimension 16)

If x !=0 then

y_i has additional terms

x_{i-1} xor x_{i-2} xor x_{i-15}

$$W_i = W_{i-3} \text{ xor } W_{i-8} \text{ xor } W_{i-14} \text{ xor } W_{i-16} \text{ xor } (W_{i-1} \text{ xor } W_{i-2} \text{ xor } W_{i-15}) >>>1$$



If y=0

$x_{i-1} \text{ xor } x_{i-2} \text{ xor } x_{i-15} = 0$

-dimension 14 code

-x_0 and x_59 independent of code

- pathological cases::
  - x_0 is non-zero and rest of x zero
  - x_59 is non-zero and rest of x zero

- it gets worse for the next column

	x’	x	y
b	a	1	0
*	1	0	0
1	0	0	0
0	0	0	0
0	0	0	0
..	..	..	..
0	0	0	0
0	0	0	0
0	0	0	0

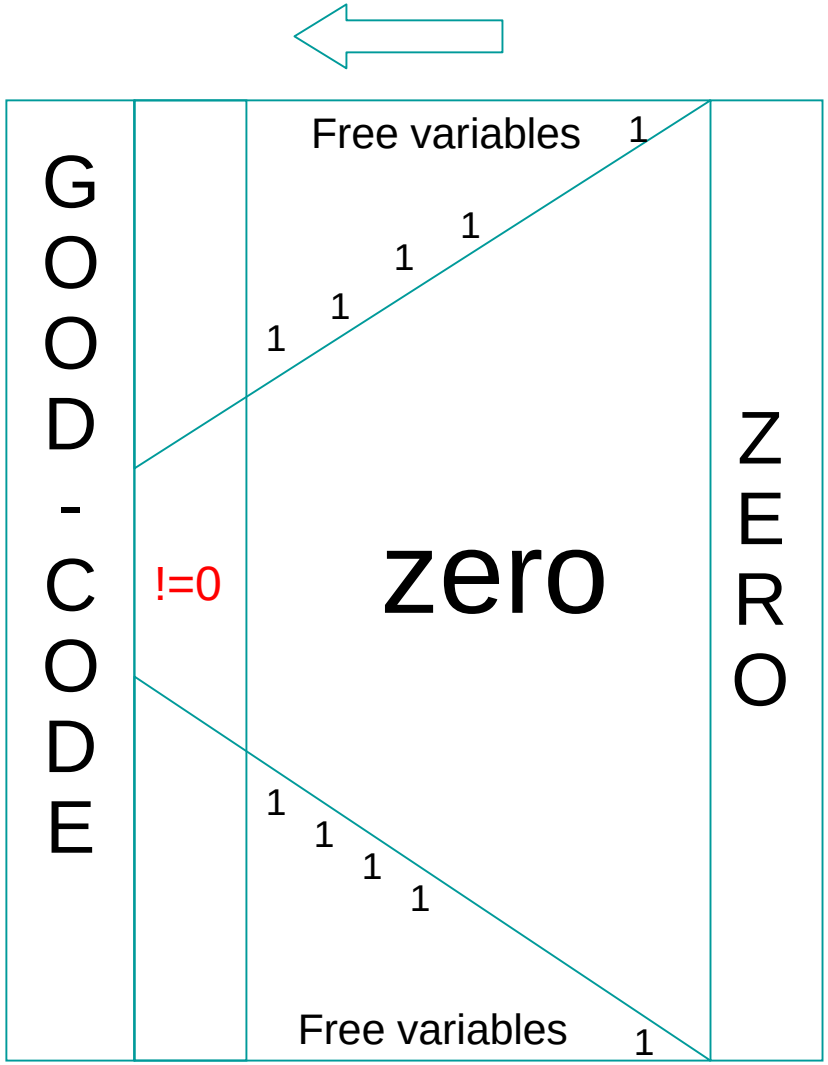
- W_i = W_{i-3} xor W_{i-8} xor W_{i-14} xor W_{i-16} xor (W_{i-1} xor W_{i-2} xor W_{i-15}) >>>1

- Thus

$$x'_{i-1} \text{ xor } x'_{i-2} \text{ xor } x'_{i-15} = x_{i-16}$$

again x’_0 is free, and if x’_{16-15}=1 rest can be zero

Thus 1+1 =2 free variables per column in pathological cases



Too many free variables

2 * #path_columns +14*3

2*8+42 = 58

SHA1-IME

$$W_i = \begin{cases} W_{i-3} \text{ xor } W_{i-8} \text{ xor } W_{i-14} \text{ xor } W_{i-16} \\ \text{xor } (W_{i-1} \text{ xor } W_{i-2} \text{ xor } W_{i-15}) <<<1, \text{ for } i = 16 \text{ to } 35 \\ \\ W_{i-3} \text{ xor } W_{i-8} \text{ xor } W_{i-14} \text{ xor } W_{i-16} \\ \text{xor } (W_{i-1} \text{ xor } W_{i-2} \text{ xor } W_{i-15} \text{ xor } W_{i-20}) <<<1, \\ \text{for } i = 36 \text{ to } 79 \end{cases}$$

Minimum weight

82 in last 64 words

At least 75 in last 60 words

At least 52 in last 48 words

TAKES 1 day of computation on 3GHZ pentium

Is this good enough?

- We prove that if a difference vector (which must be a codeword) is generated by patching together local collisions ...

  then

  the disturbance vector itself must be codeword
- We also prove more general results which allow arbitrary initial setup
- Also allows muddling in the middle
- Further muddling leads to extremely complex

  code with > 50 terms per parity check equation;
  - not systematic either

Estimated Probability of Success

- Each local collision succeeds with probability 1/4 (in the best case)
- Even if we allow conditions on messages
- Even in XOR rounds
- In MAJ rounds it is 1/16 (in best cases)
- Assuming first 32 rounds can be handled by message modifications… not likely… prob of success is at most 2^{-52* 2}

Overhead over SHA-1

- 5% software runtime overhead
- 10% hardware overhead (gate count) for high performance hardware implementations
  - comparable to SHA-256 /SHA-1 overhead
  - Some alternate codes can get rid of this overhead also
    - Needs more computationally intensive search
    - 10 days of 3GHZ pentium

Alternate Code

$$W_i = W_{i-3} \text{ xor } W_{i-8} \text{ xor } W_{i-14} \text{ xor } W_{i-16}$$
$$\text{xor}$$
$$(W_{i-1} \text{ xor } W_{i-2} \text{ xor } W_{i-11} \text{ xor } W_{i-15}) \lll 13$$
$$\text{xor}$$
$$(W_{i-1} \text{ xor } W_{i-2} \text{ xor } W_{i-11} \text{ xor } W_{i-15}) \ggg 13$$

A Simple and Provably Good Code for SHA Message Expansion

Charanjit S. Jutla
IBM Thomas J. Watson Research Center
Yorktown Heights, NY 10598
csjutla@watson.ibm.com

Anindya C. Patthak
University of Texas at Austin
Austin, TX 78712
anindya@cs.utexas.edu

Abstract

We develop a new computer assisted technique for lower bounding the minimum distance of linear codes similar to those used in SHA-1 message expansion. Using this technique, we prove that a modified SHA-1 like code has minimum distance at least 82, and that too in just the last 64 of the 80 expanded words. Further the minimum weight in the last 60 words (last 48 words) is at least 75 (52 respectively). We propose a new compression function which is identical to SHA-1 except for the modified message expansion code. We argue that the high minimum weight of the message expansion code makes the new compression function resistant to recent differential attacks.

1 Introduction

Recall the SHA-1 message expansion code: 512 information bits are packed into 16 32-bit words $\langle W_0, \cdots, W_{15}\rangle$, and 64 additional words are generated by the recurrence:

$$W_i = (W_{i-3} \oplus W_{i-8} \oplus W_{i-14} \oplus W_{i-16}) <<< 1 \quad \text{for } i \ = \ 16, \cdots, 79 \tag{1}$$

The 80 words $\langle W_0, \cdots, W_{79}\rangle$ can be seen as constituting a code-word in a linear code over $\mathbb{F}_2$ with the above parity check equations. Unfortunately, this code has a minimum distance or weight of no more than 44. Further, the weight restricted to the last 64 words is only 30. This has been exploited in [WYY05b] to give a differential attack on SHA-1 with complexity 2^{69} hash operations.

In this paper, we show that it is possible to devise codes similar to the above code of SHA-1, but with a much better minimum distance. We give a computer assisted proof that the following code has minimum distance 82, and that too in just the last 64 words:

$$W_i = \begin{cases} W_{i-3} \oplus W_{i-8} \oplus W_{i-14} \oplus W_{i-16} \oplus ((W_{i-1} \oplus W_{i-2} \oplus W_{i-15}) <<< 13) & \text{if } 16 \le i < 36 \\ W_{i-3} \oplus W_{i-8} \oplus W_{i-14} \oplus W_{i-16} \oplus ((W_{i-1} \oplus W_{i-2} \oplus W_{i-15} \oplus W_{i-20}) <<< 13) & \text{if } 36 \le i \le 79 \end{cases} \tag{2}$$

Of course, since the dimension of this code is 32×16, a brute force search of $2^{32\times 16}$ is infeasible. Thus, we have to come up with an intelligent search, and prove that all $2^{32\times 16}$ cases have been considered. Not all such codes are amenable to such a tractable search, which in our case is about 2^{48} computer instructions. Thus, we have to carefully pick the coefficients of the above parity check equations, so as to keep the search feasible and the minimum distance large.

We next propose a new variant of SHA-1, which replaces the SHA-1 message expansion code with the above code. We argue below that this leads to a compression function which is resistant

to recent differential attacks. We also argue in Section 4 that this expansion code is better than the expansion code of SHA-256, for which there is no known provable lower bound on the minimum distance. Further, in an accompanying paper [JP05b] we argue that the new variant of SHA-1 is not only resistant to recent differential attacks, but also resistant to many more natural extensions of these attacks.

A preliminary evaluation has shown that the new proposed compression function has at most a 5% overhead in speed over SHA-1 in a software implementation, and at most a 10% overhead in gate count in a high performance hardware implementation.

Recent attacks on MD-5 ([Riv92]), SHA-0 and SHA-1 (see [CJ98, BC04b, BC04a, WYY05a, WYY05b]) have capitalized on the poor message expansion of these compression functions. Essentially, all three hash functions follow the same underlying design principle: the 512-bit message is first expanded linearly into N words, and then the N words are used as step keys (sometimes known as round keys) in N steps of a (non-linear) block cipher invoked on an initial vector. The output of the block cipher is the output of the compression function.

The most effective attack against such compression functions is to launch a differential attack, where a difference in the messages leads to a zero difference in the output of the block cipher, thus leading to a collision. Unfortunately, in MD-5, SHA-0 and SHA-1, it is possible to start with a message difference which leads to a small difference in the N expanded keys. This in turn allows for a manageable overall differential characteristic of the above kind, hence leading to a collision attack.

In particular, in MD-5 a 3 bit difference in the 512-bit message leads to a difference of only 12 bits in the expanded ($N = 64$) keys. In SHA-0, there exists a message difference which leads to a 28 bit difference in the expanded ($N = 80$) keys. It turns out that the differential characteristic corresponding to the first 16 (and sometimes even first 20) steps can be assured with probability 1. Thus effectively, only the differences in latter steps contribute to lowering the probability of the differential characteristic holding. In SHA-0, the difference in the last 60 keys can be as low as 17 bits. Similarly, in SHA-1, there exists a message difference which leads to only a 27 bit difference in the last 60 keys.

Thus, the main reason that these hash functions have been undermined is their poor message expansion. With the new proposed code, any difference in messages leads to at least 82 bits of difference in the latter 64 keys. These (at least) 82 bit differences are injected into the update function of SHA-1 in the latter 64 steps, and any differential characteristic must account for canceling all (or most) of these differences. A useful heuristic that is often used in the analysis of SHA-0 and SHA-1 is that each bit difference in the key (in the latter 64 rounds) lowers the probability of success on average by a factor of $2^{2.5}$. Thus, we expect our proposed compression function to have a differential collision characteristic of probability close to $2^{-82\times 2.5}$. We also prove that the minimum weight of our proposed code in the last 60 keys is at least 75. The technique is general enough to obtain lower bounds on minimum weight of further front truncations. Note that, because of the change in the recurrence relation at $i = 36$, the codewords restricted to say the last 56 words, cannot be described as easily as the recurrence relation in Equation 2.

Organization: The rest of the paper is organized as follows: In section 2 we briefly review SHA-0, SHA-1. In section 3 we propose a new code and prove that it has good minimum distance. We then use this new code to propose SHA1-**IME**, a modified version of SHA-1. In section 4 we compare SHA1-**IME** with SHA-256 ([Uni02]) and then make a few concluding remarks.

2 SHA-0 and SHA-1

2.1 SHA-0 Message Expansion Code

In this sub-section we describe the message expansion scheme used in SHA-0. Let $\langle M_0, \cdots, M_{15} \rangle$ be the 512 bits input to SHA, where each M_i is a word of 32 bits. Then the message expansion phase of SHA-0 outputs 80 words $\langle W_0, \cdots, W_{79} \rangle$ that are computed as follows:

SHA-0 :
$W_i = M_i \quad$ for $i = 0, 1, \cdots, 15$, and

$$W_i = W_{i-3} \oplus W_{i-8} \oplus W_{i-14} \oplus W_{i-16} \qquad \text{for } i = 16, \cdots, 79. \tag{3}$$

Notice that the above can be seen as a linear code. Also notice that the expansion process applied to different bits is independent, that is there is no interleaving. This in fact makes the code rather weak and SHA-0 an easier target for the differential collision attack. Not surprisingly then that collision (and near-collision) attacks on SHA-0 have been the most successful in recent years (see [CJ98, BC04b, WYY05a]).

2.2 SHA-1 Message Expansion Code

Two years after the standard was set to SHA-0 [Uni93], an addendum was released in [Uni95], altering the message expansion scheme, and thus setting the standard to SHA-1. The change was attributed to correcting a technical weakness though no formal justification was given. The change may be interpreted as an attempt to improve the code by introducing mild interleaving. Precisely, the code in SHA-1 is the following: Let $\langle M_0, \cdots, M_{15} \rangle$ be the 512 bits input to SHA-1, where each M_i is a word of 32 bits. Then the message expansion phase outputs 80 words $\langle W_0, \cdots, W_{79} \rangle$ that are computed as follows:

SHA-1 :
$W_i = M_i \quad$ for $i = 0, 1, \cdots, 15$, and

$$W_i = (W_{i-3} \oplus W_{i-8} \oplus W_{i-14} \oplus W_{i-16}) <<< 1 \qquad \text{for } i = 16, \cdots, 79. \tag{4}$$

The notation "$<<< 1$" ("$<<< i$") denotes a one bit (i bit, respectively) rotation to the left. Note that the above code is linear too. Moreover if $\langle W_0, \cdots, W_{79} \rangle$ is a codeword, then so is $\langle W_0 <<< j, \cdots, W_{79} <<< j \rangle$ for all $j = 1, 2, \cdots, 31$. This can further be interpreted as follows: view the code-word as

$$\langle W_0^0, W_1^0, \cdots, W_{79}^0, W_0^1, \cdots, W_{79}^1, \cdots, W_{79}^{32} \rangle,$$

where W_i^j denotes the j^{th} bit of W_i. Then it is clear that this code is invariant under a rotation of 80 bits. These linear codes, a natural generalization of cyclic codes, are known as **quasi-cyclic codes** in the literature. Quasi-cyclic codes have been studied extensively over the last 40 years. (See [TW67, Che92, Lal03, LS05] and the references therein.)

Unfortunately, the interleaving process in SHA-1 is not quite good. This is observed independently in [RO05] and in [MP05]. To explain it further we rewrite Equation 4 as follows:

$$\forall i, 0 \leq i \leq 63, \quad W_i = W_{i+2} \oplus W_{i+8} \oplus W_{i+13} \oplus (W_{i+16} >>> 1), \tag{5}$$

where ">>> 1" (">>> i") denotes a one bit (i bit respectively) rotation to the right. The above clearly shows that a difference created in the last 16 words propagates to only up to 4 different bit positions. This observation allows the authors in [BC04a, RO05, MP05] to generate low-weight differential patterns. These patterns are then used to create collisions or near-collisions in reduced version of SHA-1 with complexity better than the birthday-paradox bound. Extending this further [WYY05b] reports the first attack on the full 80-step SHA-1 with complexity close to 2^{69} hash functions. In there, the authors critically observe that the code not only has small weight codewords (≤ 44, [RO05, WYY05b]) but also that these small weight codewords are even sparser in the last 60 words (for example, [WYY05b] reports a codeword with weight 27 in the last 60 words; also see [JP05a]).

3 SHA1-IME: A modified SHA proposal with a provably good code

In this section we propose a new hash function SHA1-**IME** (**IME** stands for "**I**mproved **M**essage **E**xpansion"). We use the same state update transformation as in SHA-1 or SHA-0. However, we replace the SHA-1 message expansion code by an equally simple code that has minimum distance provably at least 82, and that too in the last 64 words. The code, we denote it by $\mathcal{C}$, can be described as follows: Let $M_0, \cdots, M_{15}$ be the input message blocks. Then

SHA1-IME :
for $i = 0, 1, \cdots, 15, \quad W_i = M_i$ and
for $i = 16$ to 79

$$W_i = \begin{array}{ll} W_{i-3} \oplus W_{i-8} \oplus W_{i-14} \oplus W_{i-16} \oplus ((W_{i-1} \oplus W_{i-2} \oplus W_{i-15}) <<< 13) & \text{if } 16 \leq i < 36 \\ W_{i-3} \oplus W_{i-8} \oplus W_{i-14} \oplus W_{i-16} \oplus ((W_{i-1} \oplus W_{i-2} \oplus W_{i-15} \oplus W_{i-20}) <<< 13) & \text{if } 36 \leq i \leq 79 \end{array} \tag{6}$$

We now briefly describe the state update function used in SHA-1 (for details see [Uni95]). It comprises of total 80 steps divided in four rounds. Five 32-bits registers, conveniently denoted as A, B, C, D and E, are used. Their initial state is fixed and we denote it by $\langle A_0, B_0, C_0, D_0, E_0 \rangle$ (and in general, $\langle A_i, B_i, C_i, D_i, E_i \rangle$ after i steps). At step i, W_i is used to alter the state of these registers. Each step uses a fixed constant K_i and a bit-wise boolean function f_i that depends on the specific round. Formally,

for $i = 0$ to 79,

$$
\begin{aligned}
A_{i+1} &= W_i + (A_i <<< 5) + f_i(B_i, C_i, D_i) + E_i + K_i, \\
B_{i+1} &= A_i, \\
C_{i+1} &= B_i <<< 30, \\
D_{i+1} &= C_i, \\
E_{i+1} &= D_i,
\end{aligned}
$$

Round	Step(i)	$f_i(X, Y, Z)$
1	0-19	$XY \vee \overline{X}Z$
2	20-39	$X \oplus Y \oplus Z$
3	40-59	$XY \oplus XZ \oplus YZ$
4	60-79	$X \oplus Y \oplus Z$

where '$+$' denotes the binary addition modulo 2^{32}.

We propose the following modified version of SHA-1 : SHA1-**IME**. In the message expansion phase it uses the code described in Equation 6. Then it uses the same state update function. How does SHA1-**IME** perform compared to existing SHA-1? It is virtually the same. We used a Pentium(R) 4, 3.06 GHz machine to execute 2^{28} many hash functions. The existing SHA-1 took

```
time in sec:  567.016000, time per sha1:2.112299e-06
```

whereas SHA1-**IME** took

```
time in sec: 585.719000, time per sha2: 2.181973e-06
```

We stress that the performance of the new hash operation remains virtually the same.

3.1 Intuition behind the code

As mentioned in subsection 2.2, Equation 5 shows that the SHA-1 code does not propagate well across different bit positions. One way to remedy this situation is to let $W_i = (W_{i+2} >>> 1) \oplus W_{i+8} \oplus W_{i+13} \oplus (W_{i+16} >>> 1)$. Now Equation 4 becomes $W_i = (W_{i-3} \oplus W_{i-8} \oplus W_{i-16}) <<< 1 \oplus W_{i-14}$. Thus, whether you consider the evaluation in the *forward direction* or in the *reverse direction*, the spread of differences to the neighboring columns (i.e. neighboring bits) is more frequent. However, it is not enough to just have a good intuition about the code, but one also needs to prove a good lower bound on the minimum weight of such codes.

The strategy we use to prove lower bounds on such codes is to divide the proof into two main cases. We argue that either there are no zero columns in a codeword (a column in the codeword is the codeword projected on a particular bit position) or starting from an all zero column, the first neighboring non-zero column is actually a codeword in a good code, and so on.

Elaborating on the first case, i.e., when there are no zero columns, if every column has at least 3 bits ON, we are done. So, assume that there is some column which has 1 or 2 bits ON. Thus, there are $(64 \times 63)/2 + 64$ choices for picking these bits in the column. Having picked these bits, the neighboring column is completely specified by at most 16 bits in that column. Now the two columns together have either weight 6, in which case we are maintaining an average of 3 per column, or the weight of these two columns is at most 5. Thus, our search is quite restricted. We continue in this fashion, noting that the code has to be designed carefully so as to satisfy a property as in Claim 3.3.

As for the second case, we consider a contiguous band of zero columns, bordered on both sides with non-zero columns (we prove that they cannot be same; in fact we prove by a rank argument

that there must be at least four consecutive non-zero columns). We have to assure that when a column is zero, and the neighboring column is non-zero (whether to the right or left), the resulting code for the neighboring column is a good code, i.e., with a good minimum weight. Note that this is important since we may possibly have at most 5-6 non-zero columns. Therefore it is desired that the disturbance propagates fast across columns. Unfortunately, this is impossible for the codes we are considering so far.

Consider a SHA-1 like code, with dimension 16×32, and which is invariant under column rotations. Moreover, suppose that the code is of the form

$$W_i = \sum_{j=1}^{16} a_j W_{i-j} + \left(\left(\sum_{j=1}^{16} b_j W_{i-j} \right) <<< 1 \right),$$

where $a_1, \cdots, a_{16}, b_1, \cdots, b_{16}$ are boolean. If a_{16} and b_{16} are equal, then there is a codeword which is zero everywhere, except for W_0 which is the all 1 32-bit word. Thus for the sake of the argument, assume that $b_{16} = 0$ and $a_{16} = 1$. However in this case, suppose $j' < 16$ is the largest j such that $b_{j'}$ is non-zero. First note that if a column, say C^i, is zero, then in the column to its right, say C^{i-1}, C^{i-1}_k (for $k = 0$ to $15 - j'$) can take any value (i.e., are free variables), and the rest of the column C^{i-1} can be all zero. Further, the propagation to columns C^{i-2}, C^{i-3} etc. can be rather weak.

A similar situation arises when the code is evaluated in the backward direction. The trick is to keep the above free variables few in number, so that the subspace of such *pathological cases* is of a relatively small dimension. This small dimension is absolutely necessary to keep the exhaustive search over this space tractable. One way to get rid of these pathological free variables is to include a term like W_{i-20}, as we do in our code. This in fact gets rid of all the pathological variables in the forward direction and thereby yields a fast expansion. In the backward direction at least one pathological free variable per column remains, and we must search over such subspaces.

3.2 A lower bound on the minimum distance

In this subsection, we give a computer assisted proof to conclude that the code proposed in Equation 6 has minimum distance at least 82 in just the last 64 words. First of all observe that $\mathcal{C}$ (described in Equation 6) too is a quasi-cyclic code. To see this observe that viewed appropriately a rotation by 80 bits leaves the code invariant. Establishing lower bound on the minimum distance of a quasi-cyclic code is a hard problem and has drawn considerable attention (see [Che92, Lal03]). Unfortunately, when the index (that is the minimum amount of rotation that leaves the code invariant) is as large as 80 (or even 64), the presently known bound seems computationally infeasible. In general, it is known that computing minimum weight of an arbitrary linear code is NP-hard (see [Var97]), and that approximating within a constant factor is NP-hard under randomized reduction (see [DMS03]). An interesting approach is taken in [RO05] where they restrict their search by keeping most columns zero. This allows them to find a codeword with low weight for SHA-1; however, they do not give a technique to lower bound the minimum weight of such codes.

Secondly, observe that the code $\mathcal{C}$ in SHA1-**IME** uses a left rotation by 13 bit. However, it is easy to see that as long as the amount of rotation is relatively prime to 32, the code remains the same up to a permutation of its columns. In particular, its minimum weight does not change if left rotate by 13 is replaced by a left rotate by 1. Therefore instead of $\mathcal{C}$, we consider the following

code $\mathcal{C}'$ which is equivalent up to a permutation in the codeword positions : Let $M_0, \cdots, M_{15}$ be the message blocks. Then

for $i = 0, 1, \cdots, 15, \quad W_i = M_i$ and
for $i =$ 16 to 79

$$W_i = \begin{cases} W_{i-3} \oplus W_{i-8} \oplus W_{i-14} \oplus W_{i-16} \oplus ((W_{i-1} \oplus W_{i-2} \oplus W_{i-15}) <<< 1) & \text{if } 16 \leq i < 36 \\ W_{i-3} \oplus W_{i-8} \oplus W_{i-14} \oplus W_{i-16} \oplus ((W_{i-1} \oplus W_{i-2} \oplus W_{i-15} \oplus W_{i-20}) <<< 1) & \text{if } 36 \leq i \leq 79 \end{cases} \tag{7}$$

In fact the following explicit permutation applied to the columns in $\mathcal{C}$ yields $\mathcal{C}'$:

$$\pi : \{0, 1, \cdots, 31\} \rightarrow \{0, 1, \cdots, 31\} \text{ where } j \mapsto (5 \cdot j) \bmod 32$$

since 5 is the inverse of 13 modulo 32.

Since we will be arguing about the weight of this code in the last 64 words, we instead consider the following code $\mathcal{C}64$: Let $M_0, \cdots, M_{15}$ be the message blocks. Then

for $i = 0, 1, \cdots, 15, \quad W_i = M_i$ and
for $i =$ 16 to 63

$$W_i = \begin{cases} W_{i-3} \oplus W_{i-8} \oplus W_{i-14} \oplus W_{i-16} \oplus (W_{i-1} \oplus W_{i-2} \oplus W_{i-15}) <<< 1 & \text{if } 16 \leq i < 20 \\ W_{i-3} \oplus W_{i-8} \oplus W_{i-14} \oplus W_{i-16} \oplus (W_{i-1} \oplus W_{i-2} \oplus W_{i-15} \oplus W_{i-20}) <<< 1 & \text{if } 20 \leq i \leq 63 \end{cases} \tag{8}$$

We first prove that this is indeed sufficient.

Lemma 3.1 *If the code $\mathcal{C}64$ described above has minimum weight at least 82, then $\mathcal{C}$ has minimum weight at least 82 in its last 64 words.*

Proof: Consider any nonzero codeword in $\mathcal{C}'$, say $U = \langle U_0, \cdots, U_{79} \rangle$. Denote $X = \langle U_0, \cdots, U_{15} \rangle$ and $Y = \langle U_{16} \cdots, U_{31} \rangle$ and $Z = \langle U_{32} \cdots, U_{79} \rangle$. Therefore $U = \langle X, Y, Z \rangle$. From Equation 7 observe that the code $\mathcal{C}'$ is completely determined by specifying any consecutive 16 word block provided the block starts anywhere in 0 to 20, since the rest can then be obtained by solving the recurrence relation. We therefore choose to specify $Y = \langle U_{16}, \cdots, U_{31} \rangle$, that is we treat Y as the message symbols. Note that a fixed choice of Y also fixes X and Z. Following this observation it is now clear that $\langle Y, Z \rangle$ is a codeword in $\mathcal{C}64$.

Assume that the minimum weight of $\mathcal{C}64$ is d. Then we need to show that any non-zero codeword in $\mathcal{C}'$, has weight at least d in its last 64 words. This follows provided X being non-zero implies Y is non-zero. However, Y being zero implies X is zero, as X is a linear function of X. Therefore the minimum weight of $\mathcal{C}64$ is exactly the minimum weight of code $\mathcal{C}'$ in its last 64 words. Since $\mathcal{C}$ and $\mathcal{C}'$ is the same code up to a permutation of the co-ordinate positions, the minimum

weight of $\mathcal{C}64$ is exactly the minimum weight of code $\mathcal{C}'$ in its last 64 words. (Observe that the permutation permutes only the columns, that is i^{th} word in $\mathcal{C}$ translates into the i^{th} permuted word of $\mathcal{C}'$.) ■

Next we prove a lower bound on the minimum distance of $\mathcal{C}64$. We break down the proof into several sub-cases. In each sub-case, we argue often following an exhaustive search over a small space that the minimum weight of the code is at least 82. We mention that a naive algorithm may require to search a space as large as $2^{32\times16}$ which is clearly not feasible. Therefore the novelty in our approach lies in a careful sub-division of the problem into a small number of tractable cases. We mention that this approach is very general and may be used to give lower bounds on the minimum distance of similar quasi-cyclic codes or nearly-quasi-cyclic codes.

Theorem 3.2 *The code $\mathcal{C}64$ as defined by Equation 8 has minimum distance at least 82.*

Proof: It is easy to notice that the code $\mathcal{C}64$ is a quasi-cyclic code by noting that it is invariant under a 64 bit cyclic shift. From now onwards, we view the codewords of $\mathcal{C}64$ as a matrix that has 32 columns where each column is 64-bit long. The quasi-cyclic property then just mean that the code is invariant under column rotations. Unless otherwise specified, the arithmetic in the superscript will be modulo 32.

Now consider any non-zero codeword. Since the code is a linear code, it suffices to prove that it has weight at least 82. We break down the proof into two main cases depending upon whether or not a codeword has zero columns.

1. (**All Columns Non-Zero Case:**) Consider any such codeword. Also, consider any non-zero column, w.l.o.g., let it be C^0. Denote the columns, to the left of it by $C^1, C^2, \cdots, C^{31}$. Note that all C^i's are non-zero. In this case the following claim holds.

   Claim 3.3 *For any non-zero column C^i, there exists $k, 0 \leq k \leq 7$ such that the combined weight of columns $C^i, C^{i+1}, \cdots, C^{i+k}$ is at least $3 \cdot (k+1)$.*

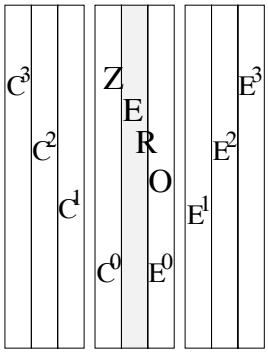
   Proof: This is easily verified by a computer program. We mention that for $k \leq 6$, an average of 3 cannot be assured (see Appendix B for an example). ■

   Next we create a partition of the 32 columns into several groups. We pick a non-zero column C^i. Now following Claim 3.3, there exists $(k+1)$-columns $(0 \leq k \leq 7)$ such that the average weight of each column is at least 3. Consider the smallest k that achieves this. Then put these $(k+1)$ columns $C^i, C^{i+1}, \cdots, C^{i+k}$ into a group. Call these columns good columns and the group a good group. We then choose C^{k+i+1} and form another group. We continue like this till no more good groups can be created. The remaining columns are then grouped together. Call this group a bad group. Note that the bad group has average weight at least 1. Now let e be the size of this bad group. Then we have $(32-e)$ good columns. Also following Claim 3.3, e could be at most 7. Therefore the total weight of the codeword is at least

   $$3 \cdot (32-e) + e = 96 - 2 \cdot e \geq 82.$$

2. (**At Least One Column Zero Case:**) Assume that there is at least one zero column. W.l.o.g. let C^0 be a zero column such that the column to the left of it is non-zero (note

that such a column always exists since we are considering a non-zero codeword). Denote the columns to the left of C^0 as $C^1, C^2, \cdots$ (see figure).



Also, going towards the right of C^0, denote the first non-zero column by E^1 and thereafter $E^2, E^3, \cdots$. Denote the column to the left of E^1 by E^0. (Note that it may be possible that C^0 and E^0 are the same column.) We argue that a few columns to the left and right of a band of zero columns must contribute a total weight of at least 82.

It will be immaterial in our analysis below if there are some non-zero columns between C^0 and E^0. All we require in our analysis is that C^0 and E^0 are zero.

Next consider $C^1, C^2, \cdots$. How soon can the sequence yield a zero column, i.e., what is the smallest value of j such that $C^j = E^0$? In order to answer this question, first note that since C^0 is everywhere zero, C^1 is essentially generated by the code whose parity check equations over $\mathbb{F}_2$ are given as follows: Denote $C^1 = \langle y_0, \cdots, y_{63} \rangle$. Then

$$\forall i, 16 \leq i \leq 63, \quad 0 = y_i + y_{i-3} + y_{i-8} + y_{i-14} + y_{i-16}. \tag{9}$$

Similarly for a fixed C^1, the column C^2 is generated by the code whose parity check equations over $\mathbb{F}_2$ are given as follows: Denote $C^2 = \langle x_0, \cdots, x_{63} \rangle$. Then

$$0 = \begin{array}{ll} x_i + x_{i-3} + x_{i-8} + x_{i-14} + x_{i-16} + y_{i-1} + y_{i-2} + y_{i-15} & \text{for } 16 \leq i \leq 19 \\ x_i + x_{i-3} + x_{i-8} + x_{i-14} + x_{i-16} + y_{i-1} + y_{i-2} + y_{i-15} + y_{i-20} & \text{for } 20 \leq i \leq 63 \end{array} \tag{10}$$

On the other hand E^1 is generated by the code whose parity check equations over $\mathbb{F}_2$ are given as follows: Denote $E^1 = \langle w_0, \cdots, w_{63} \rangle$. Then

$$0 = \begin{array}{ll} w_{i-1} + w_{i-2} + w_{i-15} & \text{for } 16 \leq i \leq 19 \\ w_{i-1} + w_{i-2} + w_{i-15} + w_{i-20} & \text{for } 20 \leq i \leq 63 \end{array} \tag{11}$$

Similarly for a fixed E^1, the column E^2 is generated by the code whose parity check equations over $\mathbb{F}_2$ are given as follows: Denote $E^2 = \langle z_0, \cdots, z_{63} \rangle$. Then

$$0 = \begin{array}{ll} w_i + w_{i-3} + w_{i-8} + w_{i-14} + w_{i-16} + z_{i-1} + z_{i-2} + z_{i-15} & \text{for } 16 \leq i \leq 19 \\ w_i + w_{i-3} + w_{i-8} + w_{i-14} + w_{i-16} + z_{i-1} + z_{i-2} + z_{i-15} + z_{i-20} & \text{for } 20 \leq i \leq 63 \end{array} \tag{12}$$

The following claim shows that at least four consecutive columns have to be non-zero.

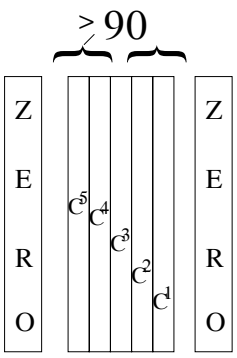
Claim 3.4 *If C^0 is everywhere zero, and C^1 is non-zero, then so is C^2, C^3 and C^4.*

Proof: Suppose for a j it is the case that $C^j = E^1$, i.e., C^{j+1} is all zero. Then a homogeneous system of linear equations over $\mathbb{F}_2$ can be set up. Consider the $64 \times j$ variables in column C^1 through C^j. There are 48 equations for each of the columns C^1 through C^j. Also, there are 48 more equations for C^{j+1}. It is well known that such a system can have a non-trivial solution if and only if the rank of the co-efficient matrix is strictly smaller than the number

of variables. It can easily be verified by a computer program that for $j = 1, 2, 3$, the system has full rank, that is exactly $64 \times j$. This can also be proved algebraically for $j = 1, 2$. We give a simple algebraic proof in the appendix (see Appendix A). ■

This proof also highlights that for the rank to be full the recurrence relation must satisfy nice properties. Ranks of all linear systems considered in this paper have been computed using Gaussian elimination. We now divide the proof into two cases.

(a) (**Number Of Consecutive Non-Zero Columns Is At Most Five**):
By the claim above, we can safely assume that we have at least four consecutive non-zero columns. Also, if we assume $C^4 = E^1$, then the number of nontrivial solutions can be at most $2^{16} - 1$ (since the co-rank or nullity of the matrix is 16, as verified by implementing a Gaussian elimination program). Similarly, assuming $C^5 = E^1$, the number of nontrivial solutions can be at most $2^{32} - 1$. We do an exhaustive search to conclude that the minimum weight in the latter case is at least 90. (Note that this latter case alone is sufficient.)

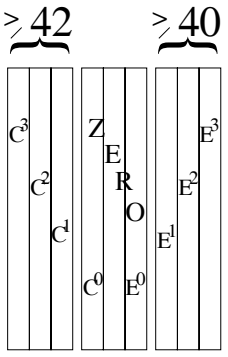


Case 2(a)

(b) (**Number Of Consecutive Non-Zero Columns Is At Least Six**): If case 1 and case 2(a) do not hold then, the only case that remains to be considered is the one where at least six consecutive columns are non-zero. Note that C^1, C^2, C^3 are then distinct from E^1, E^2, E^3. We use a computer program to verify that in this case the combined weight of C^1, C^2 and C^3 is at least 42.
Now recall Equation 11, the constraints induced on E^1. A quick observation reveals that its free variables are the first 15 bits and the very last bit. Depending on the values taken by E^1s first 15 bits we sub-divide our proof into two cases:

i. (**Non-Pathological Case:**) Here not all the first 15 bits of E^1 are zero.

This is the simpler case. In this case, the recurrence induces a good expansion. By an exhaustive search we obtain that in this case the combined weight of E^1, E^2 and E^3 is at least 40. Since the combined weight of C^1, C^2 and C^3 is at least 42, and that C^i, E^i are all distinct, together they establish this case.



Case 2(b)i

ii. (**Pathological Case:**) Here we assume that the first 15 variables of E^1 are all zero. This is the most subtle and difficult case. Going back to Equation 11, we note that in this case it must hold that $w_{63} = 1$ and for all $0 \leq i \leq 62, w_i = 0$. We call such w *pathological*.
Now consider Equation 12. We can have two cases here.
In the first case, assume that the first 15 variables of z are zero. In that case, it

must hold that $z_{62} = 1$. (Plugging in $i = 16$ to 62 in Equation 12 will yield $z_j = 0$ for all $15 \leq j \leq 61$ since $w_i = 0$ for these values.) Also note that z_{63} is free. In this case, we also call z pathological. In fact this may continue along the diagonal i.e., $E^3, E^4, \cdots$ may be pathological. If that happens then it is easy to show that the first non-zero bits of E^3 will be its 61^{st} bit, that of E^4 will be 60^{th} bit and so on. Also each column will have a free variable in its 63^{rd} bit.

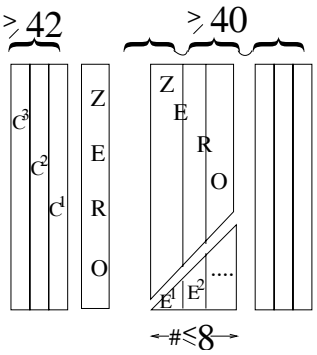
In the second case, we assume that not all of its first 15 variables are zero. We call such z's to be non-pathological.

We now sub-divide into many small cases depending primarily on the number of pathological columns (and thus on the number of free variables).

A. (**# Pathological Columns** ≤ 8) We break this case into two sub-cases. That each of these sub-cases holds has been verified using a computer program.

(I). 6^{th} **and earlier non-pathological columns are non-zero** :

In this case, we verify that the combined weight of the pathological columns and the first three non-pathological columns to the right of the pathological columns is at least 40. This ensures that in this case the minimum weight is at least 82.



Case 2(b)(ii)(A)(I)

We mention that the search space dimension can be estimated as

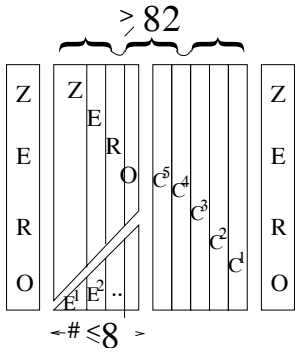
$$\#\text{ of Pathological variables} + \#\text{ of Non-Pathological Columns} \times 16,$$

which is at most 40 in this case.

We next consider the case where the non-pathological columns are same as one of C^1, C^2 or C^3.

(II). **6^{th} or earlier non-pathological column is identically zero**: Firstly note that it suffices to check the case where the 6^{th} non-pathological column is identically zero (that is $E^3 = C^3$), since other cases do fall in this case.

Now we consider the parity check equations induced on the pathological columns and the six non-pathological columns. Note that C^1 satisfies Equation 9 and that E^1 satisfies Equation 11. Also note that in between columns satisfy equations similar to Equations 10 and 12. These equations then set up a homogeneous system of linear equations whose nullity can be verified (by a computer program) to be at most 40.



Case 2(b)(ii)(A)(II)

Let the number of pathological columns be p and the number of non-pathological columns be n. Specifically then the nullity of the system can then be shown to be exactly (see Appendix A Claim A.3)

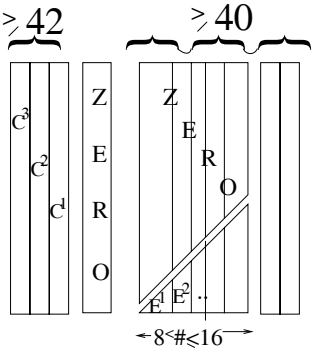
$$p + 64 \times n - 48 \times (n + 1) = p + 16 \cdot n - 48,$$

which is at most 40 in this case. We do an exhaustive search over the null space to establish that the min-weight is at least 82.

B. ($8 <$ **# Pathological Columns** ≤ 16) We also break this case into two sub-cases. That each of these sub-cases holds has been verified using a computer program.

(I). 5^{th} **and earlier non-pathological columns are non-zero**

In this case, we verify that the combined weight of the pathological columns and the first two non-pathological columns to the right of the pathological columns is at least 40. This ensures that in this case the minimum weight is at least 82.

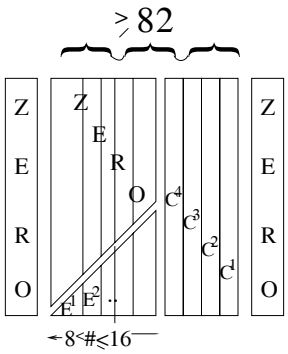


Case 2(b)(ii)(B)(I)

Therefore the case that remains to be considered is the one where the non-pathological columns are same as one of C^2 or C^3 which leads us to the next case.

(II). **5^{th} or earlier non-pathological column is identically zero**:

Firstly, note that it suffices to check the case when the 5^{th} non-pathological column is identically zero (that is $E^2 = C^3$), since other cases do fall in this case. As in the 2^{nd} sub-case of the previous case (i.e., Case 2(b)(ii)(A)(II)), we verify that the min-weight is at least 82.

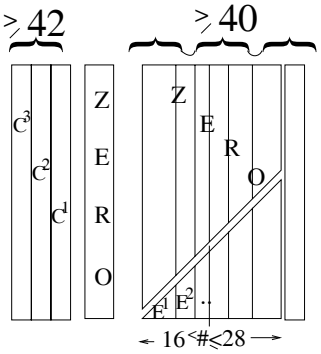


Case 2(b)(ii)(B)(II)

C. ($16 <$ **Pathological Columns** ≤ 28) First of all, notice that 28 columns is enough, since by our assumption there is at least one zero column and three non-pathological column (i.e., C^1, C^2, C^3). Now, we also break this case into two sub-cases. That each of these sub-cases holds has been verified using a computer program.

(I). 4^{th} **and earlier non-pathological columns are non-zero**

In this case, we verify that the combined weight of the pathological columns and the first non-pathological column to the right of the pathological columns is at least 40. This ensures that in this case the minimum weight is at least 82.

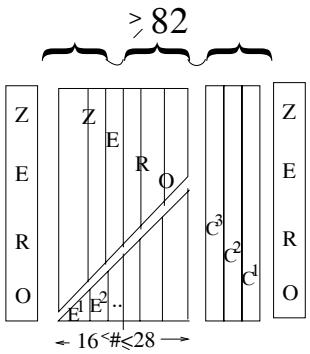


Case 2(b)(ii)(C)(I)

Therefore the case that remains to be considered is the one where the 1^{st} non-pathological column is the same as C^3.

(II). **4^{th} non-pathological column is identically zero**:

As in the 2^{nd} sub-case of the previous case (or Case 2(b)(ii)(A)(II)), we verify that the min-weight is at least 82.



Case 2(b)(ii)(C)(II)

We remark that the minimum weight of this code can at most be 82 and therefore our result is tight. We found the following codeword while searching for Case 2(b)(ii)(A)(II). Below we only give eight columns that includes six non-zero and two zero columns. The rests are all zero columns. Below the columns are placed horizontally.

```
0000000000000000 0000000000000000 0000000000000000 0000000000000000
0011110010011110 1000000001101001 1101001001010110 0000110010010000
1011000101000100 0010111101001000 1011100010101100 1101000000101111
1010101000111011 0010100100110010 1000000101001000 0110011000000000
0000000000000000 0000000000000000 0000000000000000 0000000000000100
0000000000000000 0000000000000000 0000000000000000 0000000000000011
0000000000000000 0000000000000000 0000000000000000 0000000000000001
0000000000000000 0000000000000000 0000000000000000 0000000000000000
```

3.3 The Last Sixty Words

In this subsection, we prove that the minimum weight of the code $\mathcal{C}$ in the last 60 words is at least 75. In general, our proof strategy is robust, i.e., it can in principle be adapted to estimate the minimum weight of this code in the last $4 \cdot n$ (where n is an integer) number of steps, though the dimension of the search space increases by an additive factor of $(64 - 4 \cdot n)$ and may make it computationally infeasible. On the other hand, when n gets smaller, say $n \leq 12$, we may only need

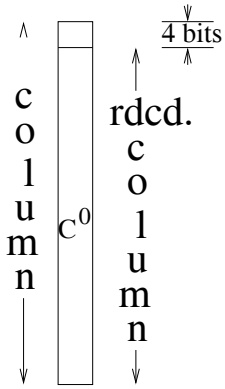
to show an average 2 per column viz a viz Claim 3.3. Since most of our search is conducted using early-stopping, the large dimension is not expected to be a problem.

Next, observe that the minimum weight of the code $\mathcal{C}64$ in the last 60 words yields a lower bound on the minimum weight of the code $\mathcal{C}$ in the last 60 words. Reviewing the proof of Theorem 3.2, it may be observed that in case 2 (i.e., **At Least One Column Zero Case**) we either consider a codeword (case 2(b)(ii)(A)(II), case 2(b)(ii)(B)(II) and case 2(b)(ii)(C)(II)) or consider few columns (in the remaining cases) which can always be extended to get a valid codeword. Therefore in these cases just counting the weight of the last 60 words gives a lower bound on the minimum weight of the code in the last 60 words. However, the same is not true for case 1 (i.e., **All Columns Non-zero Case**). We handle this case carefully. This then allows us to prove the following theorem.

Theorem 3.5 *The code $\mathcal{C}64$, as defined by Equation 8, has minimum weight at least 75 in its last 60 words.*

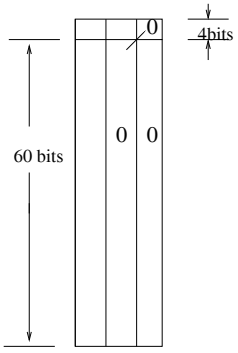
Proof: Consider any column of length 64 bits. A column restricted to its bottom most 60 bits will henceforth be referred to as a *reduced column* (see figure).

Unless otherwise mentioned, we will use the same name, eg., C^0, to denote a column and its reduced column. We divide the proof into three main cases.



A Reduced Column

1. (**All Columns Are Non-zero But Reduced Column Can Be Zero Case**): Consider any such codeword. Also consider any non-zero column, w.l.o.g., let it be C^0. Denote the columns, to the left of C^0 by $C^1, C^2, \cdots, C^{31}$. Note that by assumption all columns are non-zero.
   Then observe that due to this assumption no two consecutive reduced columns can be zero everywhere. To see this let C^0 and C^1 be the columns such that their reduced columns are everywhere zero. Let C^1 be the column left to C^0. Denote C^0 by $x = \langle x_0, x_1, \cdots, x_{63} \rangle$ and C^1 by $y = \langle y_0, y_1, \cdots, y_{63} \rangle$. Note that by the assumption $x_i = y_i = 0$ for all $i = 4, \cdots, 63$. Now consider the parity check equations of $\mathcal{C}64$ and set $i = 20$.

   We get

$$y_{20} + y_{17} + y_{12} + y_6 + y_4 + x_{19} + x_{18} + x_5 + x_0 = 0,$$

which implies $x_0 = 0$. Similarly by setting $i = 21, 22, 23$, it can be seen that x is everywhere zero.

We can therefore safely assume that no two consecutive reduced columns are zero. Then, the following can be easily verified by a computer program.

Claim 3.6 *For any non-zero column C^i, there exists $k, 0 \leq k \leq 7$ such that the combined weight of the reduced columns $C^i, C^{i+1}, \cdots, C^{i+k}$ is at least $3 \cdot (k+1)$.*

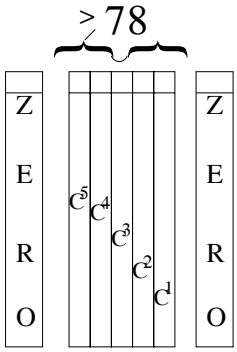
Note that although we restrict ourselves to at most 2 bits ON in reduced C^0, we must consider all 16 possibilities for the first 4 bits of C^0 to be able to define reduced column C^1 (from 16 bits in reduced column in C^1 and all the bits in C^0). Despite this the search is easily conducted.

Then, following the same line of argument as in Case 1 (**All Columns Non-Zero Case**) of Theorem 3.2, it can be shown that the total weight of the reduced columns is at least 78. This is because 25 columns yield at least 75 and the remaining seven columns yield at least 3 (since two consecutive reduced columns contribute at least 1).

2. (**At Least One Column Zero Case**): This case can be handled as the **Zero Case** in the proof of theorem 3.2. We consider the same number of cases and we count only the last 60 bits in a column. We skip the details and summarize below the results we obtain.

   (a) **Number Of Consecutive Non-Zero Columns Is At Most Five**:

   The combined weight of the 5 non-zero column is then at least 78.

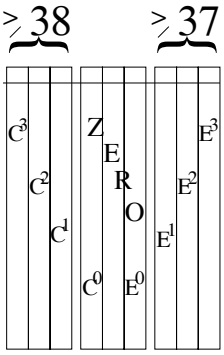


Case 3(a)

   (b) **Number Of Consecutive Non-Zero Columns Is At Least Six**: The combined weight of three reduced columns to the left of a zero band is at least 38.

      i. (**Non-Pathological Case**) The combined weight of three reduced columns to the right of a zero band is at least 38.

Therefore the combined weight of three reduced columns to the left of a zero column and that of three reduced columns to the right of a zero column yields (assuming they are distinct) at least 75.



Case 3(b)(i)

ii. (**Pathological Case**)

A. **# of Pathological columns** ≤ 8

(I). **6^{th} and earlier non-pathological columns are non-zero** : The combined weight of the pathological reduced columns and the first three non-pathological reduced columns to the right of the pathological columns is at least 37.

(II). **6^{th} or earlier non-pathological column is zero**: The combined minimum weight of these reduced columns is at least 75.

B. $8 <$ **# of Pathological columns** ≤ 16

(I). **5^{th} and earlier non-pathological columns are non-zero** : The combined weight of the pathological reduced columns and the first two non-pathological reduced columns to the right of the pathological columns is at least 37.

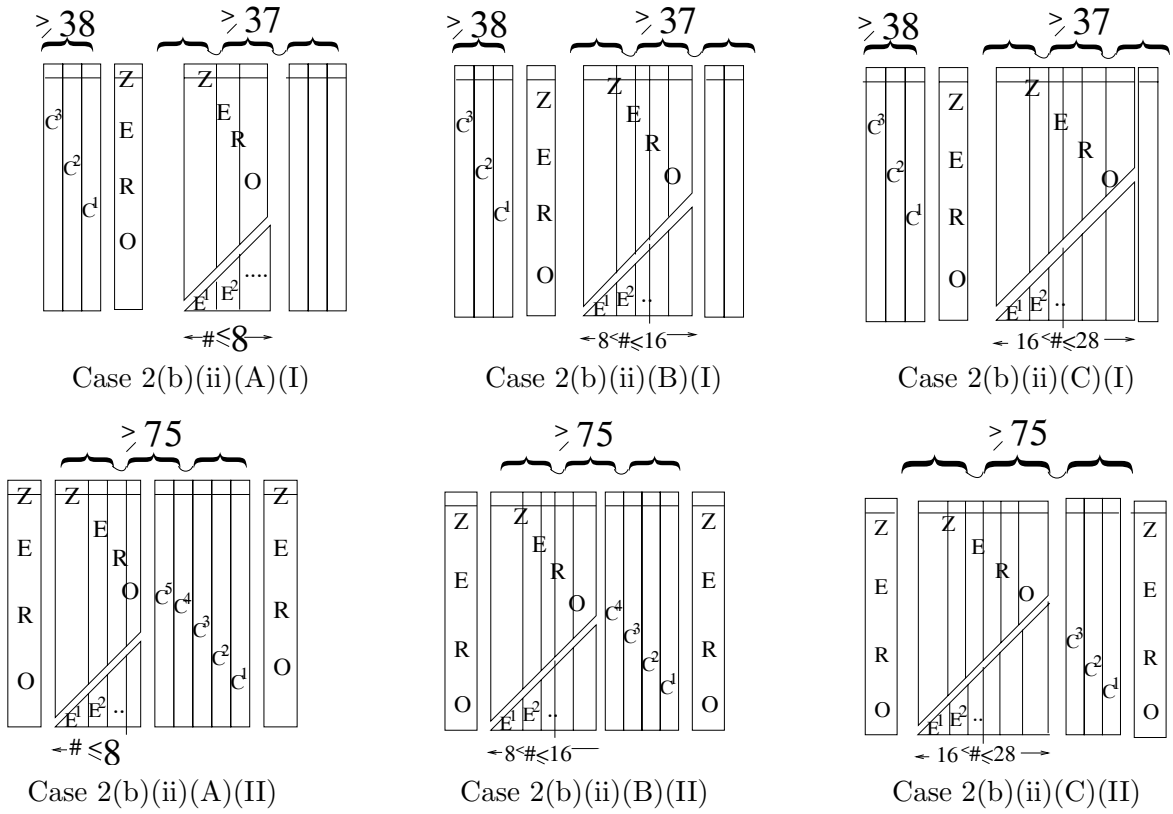
(II). **5^{th} or earlier non-pathological column is zero**: The combined minimum weight of these reduced columns is at least 75.

C. $16 <$ **# of Pathological columns** ≤ 28

(I). **4^{th} and earlier non-pathological columns are non-zero** : The combined weight of the pathological reduced columns and the first non-pathological reduced columns to the right of the pathological columns is at least 37.

(II). **4^{th} or earlier non-pathological column is zero**: The combined minimum weight of these reduced columns is at least 75.

Therefore, in all these cases the combined weight of the reduced column is at least 75. This establishes the theorem. ■



Various Cases in the proof of Theorem 3.5
(weights referred to the combined weights of the reduced columns)

Note that our result is tight. The codeword we cite in the previous subsection achieves this bound.

3.4 The Last Forty-Eight Words

In this subsection, we prove that the code $\mathcal{C}64$ has minimum weight at least 52 in its last 48 words. As mentioned previously, this proof is more computation intensive as the dimension of the search space increases by an additive factor of 16. The good thing is that we need to show an average 2 per column, viz a viz Claim 3.3. This makes our search, conducted using early-stopping, feasible in spite of the apparent large dimension.

It is easy to observe that the minimum weight of the code $\mathcal{C}64$ in the last 48 words yields a lower bound on the minimum weight of the code $\mathcal{C}$ in the last 48 words. The proof uses the same technique as in the proof of Theorem 3.5. Recall that in that proof (that is the proof of Theorem 3.5) there are cases where we either consider a codeword or consider few columns which can always be extended to get a valid codeword. In those cases, just counting the weight of the last 48 words suffices to give a lower bound on the minimum weight of the code in the last 48 words. In the remaining case, mimicking the proof of Theorem 3.5, we consider reduced columns (here restricted to last 48 entries). We then can verify that under the assumption that all columns are

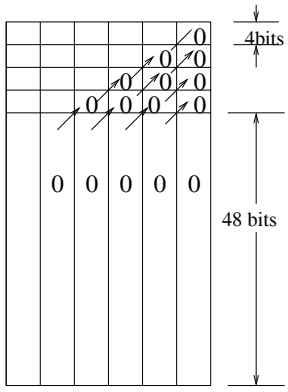
non-zero, the reduced columns cannot be too sparse. This then allows us to prove the following theorem.

Theorem 3.7 *The code $\mathcal{C}64$ as defined by Equation 8 has minimum weight at least 52 in its last 48 words.*

Proof: Consider any column of length 64 bits. Here a column restricted to its bottom most 48 bits will henceforth be referred as a *reduced column*.
Unless otherwise mentioned, we will use the same name, eg., C^0, to denote a column and its reduced column. We divide the proof into two main cases, depending on the existence of a zero column.

1. (**All Columns Are Non-Zero But Reduced Column Can Be Zero Case**): Consider any such codeword. Also consider any non-zero reduced column, w.l.o.g., let it be C^0. Denote the reduced columns, to the left of C^0 by $C^1, C^2, \cdots, C^{31}$. Note that if five consecutive reduced columns are zero, then the first column must be everywhere zero.

   This is easily obtained by setting i suitably in the parity check equations of the code $\mathcal{C}64$ (see figure). We handle that case latter. Therefore we can safely assume that no five consecutive reduced columns are zero.

   Then the following is easily verified by a computer program.

   Claim 3.8 *For any non-zero column C^i, there exists $k, 0 \leq k \leq 6$ such that the combined weight of the reduced columns $C^i, C^{i+1}, \cdots, C^{i+k}$ is at least $(k+1)$. Furthermore, there exists $\ell, 0 \leq \ell \leq 8$ such that the combined weight of the reduced columns $C^i, C^{i+1}, \cdots, C^{i+\ell}$ is at least $2 \cdot (\ell + 1)$.*

   Note that although we restrict ourselves to at most 1 bit ON in reduced C^0, we must consider all 2^{16} possibilities for the first 16 bits of C^0 to be able to define reduced column C^1 (from 16 bits in reduced column in C^1 and all the bits in C^0). Since we rely heavily on early stopping, these bits must be guessed in a lazy fashion to make the search feasible. Then following the same line of argument as in Case 1 (**All Columns Non-Zero Case**) of Theorem 3.5, it can be shown that the total weight of the reduced columns is at least 53 (since 24 columns yield at least 48 and the remaining eight columns yield at least 8, or 25 columns yield at least 50 and the remaining 7 yields 7, or 26 columns yield 52 and remaining 6 at least 1).

2. **At Least One Column Zero Case**: In this case the first column must be everywhere zero. This case can then be handled as the **Zero Case** in the proof of theorem 3.2. We consider the same number of cases and we count only the last 48 bits in a column. We remark that in each such cases, it can be shown that the weight in the last 48 rounds is at least 52. We skip the details.

4 Conclusion

4.1 Alternate codes

Notice that the code $\mathcal{C}64$ has a sliding window of size 20, that is to encode a message using this code, an LFSR would require 20 registers. The following code has a sliding of size 17. This may be useful for direct LFSR-type hardware implementation, since this would require three less registers than what the code $\mathcal{C}64$ requires.

Remark 4.1 *We mention here that using our technique, it can be shown that the following code has similar good minimum weight parameters as that of* $\mathcal{C}64$ *.*

Alternative1 *:*
for $i = 0, 1, \cdots, 15, \quad W_i = M_i$ *and*
for $i = 16$ *to 63*

$$W_i = \begin{array}{ll} W_{i-3} \oplus W_{i-8} \oplus W_{i-14} \oplus W_{i-16} \oplus ((W_{i-1} \oplus W_{i-2} \oplus W_{i-11}) <<< 13) & \text{if } 16 \leq i < 17 \\ W_{i-3} \oplus W_{i-8} \oplus W_{i-14} \oplus W_{i-16} \oplus ((W_{i-1} \oplus W_{i-2} \oplus W_{i-11} \oplus W_{i-17}) <<< 13) & \text{if } 17 \leq i \leq 63 \end{array} \tag{13}$$

We expect the following code too to have equally good properties as the codes we have considered/mentioned previously. However, because of additional pathological variables, the analysis becomes more complex and we defer the complete analysis to a later time.

Remark 4.2 $\langle W_0, \cdots, W_{79} \rangle$ *are computed from the message* $\langle M_0, \cdots, M_{15} \rangle$ *as follows:*

Alternative2 *:*
for $i = 0, 1, \cdots, 15, \quad W_i = M_i$ *and*
for $i = 16$ *to 63*

$$\begin{aligned} W_i \;=\; & W_{i-3} \oplus W_{i-8} \oplus W_{i-14} \oplus W_{i-16} \oplus ((W_{i-1} \oplus W_{i-2} \oplus W_{i-11} \oplus W_{i-15}) <<< 1) \\ & \oplus ((W_{i-1} \oplus W_{i-2} \oplus W_{i-11} \oplus W_{i-15}) >>> 1) \end{aligned} \tag{14}$$

4.2 Our proposed code vs. SHA-256 code

The code in SHA-256 ([Uni02]) is the following: Let $\langle W_0, \cdots, W_{15} \rangle$ be the 512 bits input to SHA-256, where each W_i is a word of 32 bits. Then the message expansion phase outputs $\langle W_0, \cdots, W_{63} \rangle$ where

$$\forall i, 16 \leq i \leq 63, \quad W_i \;=\; \sigma_1(W_{i-2}) + W_{i-7} + \sigma_0(W_{i-15}) + W_{i-16}, \tag{15}$$

where σ_0 and σ_1 are as follows:

$$\sigma_0(x) \stackrel{def}{=} (x >>> 7) \oplus (x >>> 18) \oplus (x >> 3).$$

$$\sigma_1(x) \stackrel{def}{=} (x >>> 17) \oplus (x >>> 19) \oplus (x >> 10);$$

In the above, "$>> i$" denotes a right shift by i bit and '+' denotes binary addition modulo 2^{32}. Note that the binary addition makes the code non-linear. We do not see how to lower bound the minimum weight of the above code. In spite of the complex description, we do not know how to formally argue about the security that this code offers.

One property that the SHA-256 code has which might be useful against [CJ98] and [WYY05b] attacks is that the code is not quasi-cyclic. These attacks require that a codeword rotated (along columns) is again a codeword. Similarly, the attacks require that the codewords shifted (along rows) is again a codeword. In fact, even our proposed code, although quasi-cyclic, is not invariant under shifts along rows. This is because the recurrence relation changes from step 36 onwards. However, claiming security on this basis maybe short-lived, and arguably there is no substitute to actually proving that the code has a high minimum weight.

4.3 Modifying SHA-256

It should be noted that SHA-256, unlike SHA-1, has only 64 steps. There are two reasons why the designers of SHA-256 probably considered it safe to reduce the number of steps: firstly, since SHA-256 produces a 128 bit output, its non-linear block cipher has eight 32 bit registers instead of the five that SHA-1 has. This in turn means that any disturbance introduced using the expanded message words W_i carries on for at least eight rounds (instead of five), and hence the probability of forcing local collisions goes down. Secondly, the SHA-256 message expansion code itself is more involved and possibly has better minimum distance (though as discussed in the previous subsection, there is no proof of that).

Utilizing the first observation, we believe that a provably good message expansion into 64 words does indeed render the "modified" SHA-256 secure against differential attacks. For the code we can use a back truncation of the code $\mathcal{C}$ analyzed in this paper, i.e. given by equation (2) but with $i <= 63$. Of course, one would need to analyze this code from scratch, as the minimum weight numbers for the code $\mathcal{C}$ do not automatically yield numbers for the back truncation.

Another interesting code, which we plan to analyze in the future, is a code similar to Alternative 2 above but with a sliding window of size 20. Recall (see the last para of section 3.1) that increasing the window size allows us to get rid of certain pathological variables, and makes the search feasible. Moreover, it also simplifies the analysis considerably. In particular, the code we plan to analyze and recommend for SHA-256 is:

$$W_i = \begin{cases} W_{i-3} \oplus W_{i-8} \oplus W_{i-14} \oplus W_{i-16} \\ \oplus((W_{i-1} \oplus W_{i-2} \oplus W_{i-15}) <<< 13) \\ \oplus((W_{i-1} \oplus W_{i-2} \oplus W_{i-15}) >>> 13) & \text{if } 16 \leq i < 36 \\ \\ W_{i-3} \oplus W_{i-8} \oplus W_{i-14} \oplus W_{i-16} \\ \oplus((W_{i-1} \oplus W_{i-2} \oplus W_{i-15}) <<< 13) \\ \oplus((W_{i-1} \oplus W_{i-2} \oplus W_{i-15} \oplus W_{i-20}) >>> 13) & \text{if } 36 \leq i \leq 63 \end{cases} \tag{16}$$

As before, we would need to lower bound its minimum weight in the last 48 (and possibly last

32) words. One crucial observation we make is that in analyzing $\mathcal{C}$, we could estimate the dimension of the subspace such that $C^5 = E^1$ (see case 2(a) in the proof of Theorem 3.2) to be about 32 (in fact exactly 32). This follows by just observing the number of variables, and the homogeneous equations involved. However, when the length of the code is reduced as above, this subspace has dimension at least 48. But, by mixing three columns at a time as in the above code, the number of equations in that case (i.e., in case 2(a) in the proof of Theorem 3.2) goes up considerably, and the null space has a more reasonable dimension of about 16.

4.4 Acknowledgment

We thank Pankaj Rohatgi for motivating us throughout this work, and particularly for his insightful comments on an earlier version of this manuscript. We also thank W. Eric Hall for providing us with the hardware estimates, and Ronald Rivest for suggesting the name SHA1-**IME**. The second author also wishes to thank Alexandar Vardy and Patrick Solé for helpful suggestions.

References

[BC04a] E. Biham and R. Chen. Near collisions of SHA-0. In *Crypto*, 2004.

[BC04b] E. Biham and R. Chen. New results on SHA-0 and SHA-1. In *Short talk presented at CRYPTO'04 Rump Session*, 2004.

[Che92] V. V. Chepyzhov. New lower bounds for minimum distance of linear quasi-cyclic and almost linear cyclic codes. In *Problems of information Transmission, Vol. 28, No. 1*, 1992.

[CJ98] F. Chabaud and A. Joux. Differential collisions in SHA-0. In *Crypto*, 1998.

[DMS03] I. Dumer, D. Micciancio, and M. Sudan. Hardness of approximating the minimum distance of a linear code. In *IEEE Transaction on Information Theory, 49(1)*, 2003.

[JP05a] Charanjit S. Jutla and Anindya C. Patthak. A Matching Lower Bound on the Minimum Weight of SHA-1 Expansion Code. Cryptology ePrint Archive, Report 2005/266, 2005. `http://eprint.iacr.org/`.

[JP05b] Charanjit S. Jutla and Anindya C. Patthak. Is SHA-1 conceptually sound? Cryptology ePrint Archive, Report 2005/350, 2005. `http://eprint.iacr.org/`.

[Lal03] K. Lally. Quasicyclic codes of index ℓ over $\mathbb{F}_q$ Viewed as $\mathbb{F}_q[x]$-submodules of $\mathbb{F}_{q^l}[x]/\langle x^m - 1\rangle$. In *Lecture Notes in Computer Science, Vol. 2643, Springer*, 2003.

[LS05] Ling and P. Solé. Structure of quasi-clcyic codes III: Generator theory. In *IEEE Transaction on Information Theory*, 2005.

[MP05] K. Matusiewicz and J. Pieprzyk. Finding good differential patterns for attacks on SHA-1. In *International Workshop on Coding and Cryptography*, 2005.

[Riv92] R. Rivest. RFC1321: The MD5 message-digest algorithm. In *Internet Activities Board*, 1992.

[RO05] V. Rijmen and E. Oswald. Update on SHA-1. In *Lecture Notes in Computer Science, Vol. 3376, Springer*, 2005.

[TW67] R. L. Townsend and E. J. Weldon. Self-orthogonal quasi-cyclic codes. In *IEEE Transaction on Information Theory*, 1967.

[Uni93] United States Department of Commerce, National Institute of Standards and Technology, Federal Information Processing Standard Publication #180. *Secure Hash Standard*, 1993.

[Uni95] United States Department of Commerce, National Institute of Standards and Technology, Federal Information Processing Standard Publication #180-1 (addendum to [Uni93]). *Secure Hash Standard*, 1995.

[Uni02] United States Department of Commerce, National Institute of Standards and Technology, Federal Information Processing Standard Publication #180-2. *Secure Hash Standard*, August, 2002.

[Var97] A. Vardy. The intractability of computing the minimum distance of a code. In *IEEE Transaction on Information Theory, 43(6)*, 1997.

[WYY05a] X. Wang, H. Yu, and Y. L. Yin. Efficient collision search attacks in SHA-0. In *Crypto*, 2005.

[WYY05b] X. Wang, H. Yu, and Y. L. Yin. Finding collisions in the full SHA-1. In *Crypto*, 2005.

A Rank proofs

Claim A.1 *If C^0 is zero, and C^1 is non-zero, then C^2 is non-zero.*

Proof: Assume otherwise i.e., that C^2 is zero. Consider the following 48×64 dimensional parity check matrices (essentially Equations 9 and 11) over $\mathbb{F}_2$

$$\begin{pmatrix} 1010000010000100100000 & \cdots & 000000000000000000 \\ 0101000001000010010000 & \cdots & 000000000000000000 \\ \ddots & \cdots & \ddots \\ 000000000000000000000 & \cdots & 010100000100001001 \end{pmatrix}$$

H_1

$$\begin{pmatrix} 01000000000000011000000 & \cdots & 00000000000000000000000 \\ 00100000000000001100000 & \cdots & 00000000000000000000000 \\ 00010000000000000110000 & \cdots & 00000000000000000000000 \\ 00001000000000000011000 & \cdots & 00000000000000000000000 \\ & & \\ 10000100000000000001100 & \cdots & 00000000000000000000000 \\ 01000010000000000000110 & \cdots & 00000000000000000000000 \\ \ddots & \cdots & \ddots \\ 00000000000000000000000 & \cdots & 10000100000000000000110 \end{pmatrix}$$

H_2

Then we need to show that $H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$ has full rank. To do that it is enough to show that there are 64 linearly independent rows. We consider the 48 rows of H_1 and 16 additional rows, namely 5^{th} through 20^{th} rows of H_2. We reduce the problem to showing that a certain equation over polynomial ring $\mathbb{F}_2[x]$ does not have solutions in a restricted set of polynomials. We associate with the vector $c = \langle c_0, \cdots, c_{63} \rangle$ in $\mathbb{F}_2^{64}$ the polynomial $c(s) = \sum_{i=0}^{63} c_i s^i$ in $\mathbb{F}_2[s]$. Then the following polynomials can be associated with the 1^{st} and 5^{th} rows of matrix H_1 and H_2, respectively:

$$p(s) \stackrel{def}{=} s^{16} + s^{13} + s^8 + s^2 + 1,$$

$$r(s) \stackrel{def}{=} s^{19} + s^{18} + s^5 + 1.$$

Further note that the i^{th} (note $1 \leq i \leq 48$) row of H_1 then gets associated with $s^{i-1}p(s)$. Similarly the j^{th} (note we restrict ourselves to $5 \leq j \leq 20$) row of H_2 then gets associated with $s^{j-5}r(s)$. Therefore, observe that if the 80 rows that we are considering were dependent then we can translate that to a non-zero solution of the following polynomial equation:

$$p(s)\alpha(s) + \beta(s)r(s) = 0,$$

with additional constraints that degree(α) ≤ 47 and degree(β) ≤ 15. However, it is well known that $p(s)$ is irreducible, therefore if such a equation holds then it must be the case that $p(s)$ divides $r(s)$. However, it is easy to check that $p(s)$ does not divide $r(s)$, thus leading to a contradiction. Therefore H has full rank. ■

Claim A.2 *If C^0 is zero, and C^1 is non-zero, then C^2, C^3 is non-zero.*

Proof: Consider the following polynomials :

$$p(x) \stackrel{def}{=} x^{16} + x^{13} + x^8 + x^2 + 1,$$

$$q(x) \stackrel{def}{=} x^{15} + x^{14} + x,$$

$$r(x) \stackrel{def}{=} x^{19} + x^{18} + x^5 + 1 = x^4 \cdot q(x) + 1.$$

Let H_1 and H_2 be as above.

First of all note that H_2 has full rank. (This is clear from the matrix. Otherwise, note that we could have an identity

$$q(x) \cdot a(x) + r(x) \cdot b(x) \ = \ 0$$

with degree$(a) \leq 3$ and degree$(b) \leq 43$. Since degree$(q \cdot a) <$ degree(r), this cannot happen.) Now we will show that the rank of the matrix

$$\begin{pmatrix} H_2 & 0 \\ H_1 & H_2 \\ 0 & H_1 \end{pmatrix}$$

is at least 128. Since H_1 has full rank, observe that

$$\begin{pmatrix} H_1 & H_2 \\ 0 & H_1 \end{pmatrix}$$

has rank at least 96. So consider the following 92 independent rows from the above matrix, namely 5^{th} row onwards. We also argue that another additional 5^{th} through 40^{th} rows of the top H_2 are also independent. If not, then they would satisfy the following polynomial equations

$$\begin{aligned} \alpha(x)p(x) + \beta(x)r(x) &= 0 \qquad (17) \\ x^4\beta(x)p(x) + \gamma(x)r(x) &= 0 \qquad (18) \end{aligned}$$

with restrictions degree$(\alpha) \leq 47$, degree$(\beta) \leq 43$, and degree$(\gamma) \leq 35$.

Since $p(x)$ is an irreducible polynomial, and $p(x) \nmid r(x)$, observe from Equation 17 that $p(x)|\beta(x)$. Hence, set $\beta(x) = \mu(x)p(x)$. Substituting in Equation 18 we get

$$x^4 p(x)^2 \mu(x) + \gamma(x)r(x) = 0.$$

Since $p(x)$ is irreducible, and $p(x) \nmid r(x)$, and $x \nmid r(x)$, it must hold that $x^4 p(x)^2 | \gamma(x)$. But that is impossible, since degree$(\gamma) \leq 35 <$ 36 =degree$(x^4 p(x)^2)$. ■

Recall that we used E^0 to denote a column that is zero everywhere. Also, recall that the columns left to E^0 are denoted E^1, E^2 and so on. In the following claim, we will assume $3 \leq n$.

Claim A.3 *Let $E^1, E^2, \cdots, E^p$ be p pathological columns. Also, let $E^{p+1}, E^{p+2}, \cdots, E^{p+n}$ be n non-pathological columns. Further assume that $E^{p+n+1} = C^0$ is everywhere zero. If the nullity of the parity check equations resulting from these columns with $p = 0$ is $16 \cdot n - 48$, then the nullity of the parity check equations resulting from these columns with any $p \leq 28$ is*

$$p + 16 \cdot n - 48.$$

Proof: Let $N_{i,j}, (1 \leq i \leq n, 0 \leq j \leq 63)$ denote the entries in the non-pathological columns. Also let $P_{i,j}, (1 \leq i \leq p,$ for each $i,\ 64 - i \geq j \leq 63)$ be the pathological variables. We will denote $N_i = \langle N_{i,0}, \cdots, N_{i,63} \rangle$ and $P_i = \langle P_{i,64-i}, \cdots, P_{i,63} \rangle$. Let $H_{1|i}$ denote the matrix H_1 restricted to the last i columns. (Note that only the last i rows will be non-zero.) Also let $H_{2|i}$ denote the matrix H_2 restricted to the last i columns. (Note that only the last $i-1$ rows will be non-zero.) Note that

$\langle P_1, \cdots, P_p, N_1, \cdots, N_n \rangle$ must belong to the null space of the following matrix:

$$\mathcal{H} = \begin{pmatrix} H_{1|1} & H_{2|2} & & & & & & & & & \\ & H_{1|2} & H_{2|3} & & & & & & & & \\ & & & \ddots & \ddots & & & & & & \\ & & & & & H_{1|p-1} & H_{2|p} & & & & \\ & & & & & & H_{1|p} & H_2 & & & \\ & & & & & & & H_1 & H_2 & & \\ & & & & & & & \ddots & \ddots & & \\ & & & & & & & & & H_1 & H_2 \\ & & & & & & & & & & H_1 \end{pmatrix}$$

Note that when we restrict H_1 or H_2 to the last few columns, the top rows in that restricted entries may become zero row. We remove such rows if the entire row in the above matrix $\mathcal{H}$ becomes everywhere zero. Note that with this modification, the following sub-matrix is already in the echelon form:

$$\mathcal{H}1 = \left.\begin{pmatrix} H_{1|1} & H_{2|2} & & & & \\ & H_{1|2} & H_{2|3} & & & \\ & & & \ddots & \ddots & \\ & & & & H_{1|p-1} & H_{2|p} \end{pmatrix}\right\} (p-1) \text{ blocks}$$

(Observe that first block corresponding to $(H_{1|1}\ H_{2|2})$ reduces to (1 10), and that corresponding to $(H_{1|2}\ H_{2|3})$ reduces to $\begin{pmatrix} 10 & 100 \\ 01 & 110 \end{pmatrix}$.)

Furthermore, since by assumption the following sub-matrix has full rank:

$$\mathcal{H}2 = \left.\begin{pmatrix} H_2 & & & \\ H_1 & H_2 & & \\ \ddots & \ddots & & \\ & & H_1 & H_2 \\ & & & H_1 \end{pmatrix}\right\} (n+1) \text{ blocks}$$

the matrix $\mathcal{H}$ has full rank. Note here that in the top $48 - p$ rows, $H_{1|p}$ is entirely zero. However these rows in $\mathcal{H}$ are independent since $\mathcal{H}2$ has full rank. In the remaining rows $H_{1|p}$ is in echelon form and hence independent. Note that it has number of rows i.e., constraints:

$$48 \times (n+1) + \sum_{i=1}^{p-1} i = 48(n+1) + \frac{p(p-1)}{2}.$$

Also, note the number of variables i.e., columns is

$$64 \times n + \sum_{i=1}^{p} i = 64 \cdot n + \frac{p(p+1)}{2}.$$

Thus the nullity of the system is

$$64 \cdot n + \frac{p(p+1)}{2} - \left(48(n+1) + \frac{p(p-1)}{2}\right) = p + 16 \cdot n - 48.$$

This completes the proof.

■

B Examples

We cite below an example where over 7 columns an average of 3 does not hold. Below we only give 8 columns and the columns are placed horizontally. Note that the 8 columns yield 29, whereas the first 7 columns yield only 14.

```
00000000000000000000000000000000000000000000000000000001000000
00000000000000000000000000000000000000000000000000000000110110
00000000000000000000000000000000000000000000000000000000010100
00000000000000000000000000000000000000000000000000000000001110
00000000000000000000000000000000000000000000000000000000000100
00000000000000000000000000000000000000000000000000000000000011
00000000000000000000000000000000000000000000000000000000000001
10001010100000000010010000100000100001001011000000100010000010000
```

Is SHA-1 conceptually sound?

Charanjit S. Jutla
IBM Thomas J. Watson Research Center
Yorktown Heights, NY 10598
csjutla@watson.ibm.com

Anindya C. Patthak
University of Texas at Austin
Austin, TX 78712
anindya@cs.utexas.edu

Abstract

We show that if a collision in SHA-1 is obtained as in the Chabaud and Joux attack (which is also the basis for Wang et al attack) by putting together local collisions from step t onwards, then the disturbance vector must be a projection of a codeword from step $t+5$ onwards. We conclude that SHA1-IME (proposed by Jutla and Patthak) is resistant to all recent differential attacks and their natural extensions, as the code has minimum weight 75 (52) even when restricted to the last 60 (48 respectively) steps.

1 Introduction

We briefly recall the message expansion code and the state update transforms of SHA-1 [Uni95]. Let $M_0, \cdot\ \cdot, M_{15}$ be the input message blocks. Then

SHA-1 :
for $i\ =\ 0, 1, \cdot\ \cdot, 15, \quad W_i = M_i$ and
for $i\ =\ 16$ to 79

$$W_i = (W_{i-3} \oplus W_{i-8} \oplus W_{i-14} \oplus W_{i-16}) <<< 1 \tag{1}$$

where $<<< 1$ denotes a one bit rotation to left. The state update functions are given as follows:
for $i = 0$ to 79:

$A_{i+1}\ =\ W_i + (A_i <<< 5) + f_i(B_i, C_i, D_i) + E_i + K_i,$
$B_{i+1}\ =\ A_i,$
$C_{i+1}\ =\ B_i <<< 30,$
$D_{i+1}\ =\ C_i,$
$E_{i+1}\ =\ D_i,$

Round	Step(i)	$f_i(X, Y, Z)$
1	0-19	$XY \vee \overline{X}Z$
2	20-39	$X \oplus Y \oplus Z$
3	40-59	$XY \oplus XZ \oplus YZ$
4	60-79	$X \oplus Y \oplus Z$

where '$+$' denotes the binary addition modulo 2^{32}. The output of the compression function is the string of five words $A_{80}, B_{80}, C_{80}, D_{80}$ and E_{80}.

Before we go into the collision resistance properties of SHA-1 and SHA1-**IME** [JP05], lets explore their properties as one-way functions. Specifically, it is desired that no program should be able to solve for m in reasonable time and with high probability, given a random output $\langle \alpha, \beta, \gamma, \mu, \nu \rangle$, i.e.

$$A_{80}(m) \;=\; \alpha \wedge B_{80}(m) \;=\; \beta \wedge C_{80}(m) \;=\; \gamma \wedge D_{80}(m) \;=\; \mu \wedge E_{80}(m) = \nu$$

It is widely believed that the above property of one-wayness holds for SHA-1, and we briefly describe the heuristic argument which is used in making this assumption. Before that, we must note that actually proving the above one-way claim, even in an asymptotic sense is an extremely difficult problem. One approach could be to show that (in an asymptotic version of the above problem) the problem can be framed as a Polynomial Constraint Satisfaction Problem over $\mathbb{F}_2$ (where each polynomial has degree at most two), which is known to be NP-hard ([GJ79]). However, the notion of cryptographic one-wayness requires showing the problem to be average-case hard for NP. Unfortunately, all advances in this direction have been stymied by a theorem of Impagliazzo ([Imp95]) that any such result must be non-relativizing. Further, it has been shown ([FF76, BT05]) that under non-adaptive reductions this reduction is not possible unless the polynomial hierarchy collapses to the third level (of course, there could still be adaptive reductions, but then the proof must be non-relativizing). The other approach requires showing super-polynomial lower bounds directly on the average case complexity of the given problem, for which all known techniques are inadequate [RR97].

Coming back to the heuristic argument for making the case for SHA-1 as a one-way function, let us take a look at the actual equations or constraints involved. Recall, a program P must solve for m given $\alpha, \beta, \gamma, \mu, \nu$.

We denote the j^{th} bit of word X_i by $x_{i,j}$. Also, when we consider any addition of two variables we will introduce binary auxiliary variables to represent carries, and we will use Greek letters to denote them. Also, we break up any addition involving more than two words into several two word additions, thus ensuring that carry variables are binary.

$\forall i \in [1, 80]\ \forall j \in [0, 31]$

$$\begin{aligned}
f_{i,j} &= f_i(a_{i-2,j}, a_{i-3,j+2}, a_{i-4,j+2}) \\
g_{i,j} &= (k_{i,j} + w_{i-1,j} + \sigma_{i,j-1}) \bmod 2 \\
\sigma_{i,j} &= (k_{i,j} + w_{i-1,j} + \sigma_{i,j-1}) \text{ div } 2 \\
h_{i,j} &= (g_{i,j} + f_{i-1,j} + \tau_{i,j-1}) \bmod 2 \\
\tau_{i,j} &= (g_{i,j} + f_{i-1,j} + \tau_{i,j-1}) \text{ div } 2 \\
m_{i,j} &= (h_{i,j} + a_{i-5,j+2} + \phi_{i,j-1}) \bmod 2 \\
\phi_{i,j} &= (h_{i,j} + a_{i-5,j+2} + \phi_{i,j-1}) \text{ div } 2 \\
a_{i,j} &= (m_{i,j} + a_{i-1,j-5} + \psi_{i,j-1}) \bmod 2 \\
\psi_{i,j} &= (m_{i,j} + a_{i-1,j-5} + \psi_{i,j-1}) \text{ div } 2
\end{aligned} \tag{2}$$

with $\sigma_{i,-1} = \tau_{i,-1} = \phi_{i,-1} = \ _{i,-1} = 0$ for all i.

$$\begin{aligned}
\forall j \in [0,31]\ a_{0,j} &= IV_{0,j} \\
\forall j \in [0,31]\ b_{0,j} &= a_{-1,j} = IV_{1,j} \\
\forall j \in [0,31]\ c_{0,j} &= a_{-2,j} = IV_{2,j} \\
\forall j \in [0,31]\ d_{0,j} &= a_{-3,j} = IV_{3,j} \\
\forall j \in [0,31]\ e_{0,j} &= a_{-4,j} = IV_{4,j}
\end{aligned} \tag{3}$$

where IV_i ($i \in [0..4]$) is the initial vector specified in SHA-1.

We remark that three binary variables added and divided by two is just the majority function over GF2, and three binary variables added and reduced modulo two is just addition in GF2. Thus, all of the above equations, including the message expansion, and the computation of f, can be written as polynomial equations over GF2, with degree at most two. However, this variant of the polynomial constraint satisfaction problem is known to be NP-hard ([GJ79]). The heuristic argument we employ is that this system of polynomial constraints is general enough (on average for randomly chosen $\alpha, \beta, \gamma, \mu$ and ν), and even though this is a fixed input length problem, the number of polynomial equations involved is large enough.

To further convince oneself that there is no useful structure in this system of polynomial equations, consider the above equations with all message words set to zero (and hence all W variables set to zero). Then considered over 32 bit words, there is a unique output, i,e. $A_{80}, B_{80}, ..., E_{80}$, and hence only 1 in 2^{160} of the $\alpha, \beta, ..., \nu$ will satisfy the above constraints. If we throw in the M variables (and the intermediate W variables), any attempt to solve for M (given $\alpha, \beta, ..., \nu$) is thwarted by the fact that each m_i is used in several equations, and in no particular or discernible fashion. This can be attributed to the "orthogonal" design of the state update function and the message expansion code. The additive constants K_i, and a non-zero IV make sure that the equations are not homogeneous.

Can a similar heuristic argument be given for collision resistance? We address this question in the next section. We end this section by an attempt to solve the system of equations using Schoning's Algorithm for 3-SAT and CSPs [Sch99]. Recall that in Schoning's algorithm, a random initial assignment of the n variables is chosen, and if some clause is falsified, a random literal from the clause is picked and flipped. This process is continued for up to $3n$ steps, before a new random assignment is picked. The complexity of this randomized algorithm is within a polynomial factor of $(2(1 - \frac{1}{k}))^n$, where k is the number of literals per clause.

In our case, we have written our system of equations or constraints as 4-constraints. However, the number of variables has shot up considerably from 160. Even, if we consider auxiliary variables other than W to be not a factor, we still have a total of 80×32 variables, out of which we may fix at most $512 - 160$ variables corresponding to M, thus leaving us with at least $n = 64 \times 32$ variables.

If we get rid of the W variables as well, and directly write the clauses in terms of M, we find that each clause has on average 80 binary variables corresponding to M, rendering k to be large. Note that the same argument holds for SHA1-IME.

2 Collision Resistance Properties of SHA1-IME

The hash function SHA-IME [JP05], is exactly the same as SHA-1 except that the message expansion code as in equation (1) is replaced by

$$W^i = \begin{cases} W^{i-3} \oplus W^{i-8} \oplus W^{i-14} \oplus W^{i-16} \oplus \left(W^{i-1} \oplus W^{i-2} \oplus W^{i-15}\right) <<< 13 & \text{if } 16 \leq i < 36 \\ W^{i-3} \oplus W^{i-8} \oplus W^{i-14} \oplus W^{i-16} \oplus \left(W^{i-1} \oplus W^{i-2} \oplus W^{i-15} \oplus W^{i-20}\right) <<< 13 & \text{if } 36 \leq i \leq 79 \end{cases} \tag{4}$$

It was shown that this code has minimum weight 80 in just the last 64 words. Further, the minimum weight restricted to the last 60 (48 steps) is at least 75 (52 respectively).

The attacker's task is to find two messages M and M' which hash to the same value, say $\alpha, \beta, \gamma, \mu$ and ν. If we use primed variables to write another set of equations as in the previous section for the second message M', then we can get rid of α, β etc. by equations of the form $a_{80,j} = a'_{80,j}$, $a_{79,j} = a'_{79,j}$ etc.

We will use the prefix Δ to denote the xor difference of a variable with its primed variable; thus $\Delta a_{65,12}$ denotes $a_{65,12} \oplus a'_{65,12}$. Thus, the equations in the previous paragraph are really

$$\Delta a_{80,j} = 0,\ \Delta a_{79,j} = 0, ...,\ \Delta a_{76,j} = 0 \tag{5}$$

Is it possible to write all the equations in terms of difference variables? For this to be true, exclusive-or has to distribute over the majority function which, although not always true, does happen with non-trivial probability. This leads to a trivial solution, i.e. $\Delta M = 0$, which is not very useful. However, many equations can be made to be trivially true (and with probability one) if the difference variables involved are zero. One then tries to focus on a non-zero ΔM which requires the least number of equations to have to go through the probabilistic distribution of xor over majority.

This is the idea behind local collisions [CJ98], [WYY05], as with local collisions one takes a non-zero ΔM and tries to set most of the difference variables to zero as quickly as possible. Notice that a non-zero ΔM leads to a non-zero ΔW, which keeps disturbing the equality of intermediate step variables. This disturbance is then offset as quickly as possible by additional differences coming from ΔW.

Other than this local collision strategy, or a method which makes many equations true with high probability, it is an open problem to find collisions in a way better than birthday attack. The only alternative seems to be to use general purpose randomized algorithms for satisfiability of CSPs like Schoning's algorithm mentioned in the previous section.

So, at the present state of knowledge about solving general CSPs, and without any further insights into any special structure these CSPs may have, an attacker is left with the option of trying to optimize the local collision based attacks.

In the next section we show that, if the linear message expansion code is good then either the disturbance vector itself has a large hamming weight, or finding a small hamming weight disturbance vector (if any such exists) is an instance of a NP-hard problem (which does not seem to have any special structure, e.g. sparsity, to reduce its complexity).

3 Are there better Disturbance vectors?

Recall that in [CJ98] a *disturbance vector* indicates where new disturbances start, and these individual disturbances are cancelled in the next six rounds by additional differences in the expanded message; each of these events is called a *local collision*. The linear combination (xor) of all these disturbances and additional differences is called the *difference vector*. There can be many kinds of local collision strategies, and we explore all such possibilities in the next few sub-sections. The one requirement on the difference vector is that it must be a codeword of the SHA-1 message expansion linear code.

To review recent attacks, in [CJ98] the collision attack on SHA-0 is carried out by explicitly constructing a difference vector out of a disturbance vector. Moreover, in there a disturbance vector is itself chosen to be a codeword of SHA-0. Later [BC04] and [RO05] extend their technique to cause collisions in reduced SHA-1. In [WYY05], this idea is further extended to attack the full SHA-1. However, the difference vector is still constructed out of a disturbance vector which is a codeword. One advance has been in not requiring local collisions at every disturbance – in particular, the first 16 to 20 rounds in [WYY05] are handled in a more complicated juxtaposition of local collisions, and the last 5 round disturbances are allowed to run loose. The inner round (i.e. from round 20 to round 75) disturbances however are still handled by local collisions.

Thus, since these attacks crucially depends on the weight of the disturbance vector (particularly in the inner rounds), the issue of obtaining a small weight disturbance vector is a crucial one.

3.1 Local Collision Based Strategies

In this sub-section, we prove that if the local collision is constructed as in [CJ98], then the disturbance vector must be a codeword in SHA-1 (the same is also true for SHA1-IME). It was an open problem whether one could consider a disturbance vector which is not a codeword and yet the difference vector is a codeword. To be specific, we show that if the global collision arises as a result of local collisions from step t onwards (that is we allow one to manipulate the disturbances and additional differences in any arbitrary way till the first t steps) then the disturbance vector restricted to steps $t+5$ onwards must be a SHA-1 message expansion codeword.

Consider the front-truncated SHA-1 code, i.e. restricted to the last 65 words. We assume that the parity check constraints of SHA-1 is denoted by R, i.e., for any 65×32-bit vector w, $R(w) = 0$ iff

$$\forall j \in \{1, \quad , 32\}, \forall i \in 16, \quad , 64 \;\; w_{i,j} \;=\; w_{i-3,j-1} \oplus w_{i-8,j-1} \oplus w_{i-14,j-1} \oplus w_{i-16,j-1}.$$

Assume that the local collisions are desired from rounds 5 to 64. 10 Also, we do not force that the last 5 words in the disturbance vectors be zero, thus allowing the possibility of near-collisions.

Define a map $\imath : \{0,1\}^{2^{65\times 32}} \to \{0,1\}^{2^{65\times 32}}$. Let $z \stackrel{def}{=} \imath(u)$. Then

$$\imath(u)_{i,j} \;=\; z_{i,j} \stackrel{def}{=} \begin{cases} u_{i,j} & \text{if } 0 \le i \le 4 \\ u_{i,j} \oplus u_{i-1,j-5} \oplus u_{i-2,j} \oplus u_{i-3,j+2} \oplus u_{i-4,j+2} \oplus u_{i-5,j+2} & \text{if } 5 \le i \le 64 \end{cases} \tag{6}$$

Essentially, the map $\imath$ takes a disturbance vector u, and builds a difference vector $\imath(u)$ according to a local collision strategy for the last 60 rounds, whereas in the first 5 rounds the difference vector is already specified by the disturbance vector, and hence need not follow any particular pattern. We will call this map, or this local collision strategy a **CJ-local collision strategy** [CJ98].

The main result of this section shows that if a difference vector z is obtained from a disturbance vector u by the above transformation, i.e. $z = \imath(u)$, and that z is a codeword (as every difference vector must be) then u, the disturbance vector, agrees with some codeword y in the last 60 rounds. The import of this result is that the disturbance vector cannot be small weight (in SHA1-IME) if a global collision is obtained by patching together local collisions, even when allowing for freedom in the first 20 or so rounds.

Let $y_{i,j}$ be any 65×32 (where $0 \le i \le 64, 0 \le j \le 32$) bit vector. We also use y_i to denote the vector $y_i \stackrel{def}{=} \langle y_{i,j} \rangle_{j=0}^{31}$. Given y, we define another map $\rho : \{0,1\}^{2^{65\times 32}} \to \{0,1\}^{2^{65\times 32}}$. Let $u \stackrel{def}{=} \rho(y)$. Then

$$\rho(y)_{i,j} \;=\; u_{i,j} \stackrel{def}{=} \begin{cases} y_{i,j} & \text{if i} \ge 5 \\ y_{i,j} \oplus y_{i-1,j-5} \oplus y_{i-2,j} \oplus y_{i-3,j+2} \oplus y_{i-4,j+2} \oplus y_{i-5,j+2} & \text{if } 0 \le i \le 4 \end{cases} \tag{7}$$

where y_{-1} to y_{-5} are obtained from y by the SHA-1 expansion code run backwards.

Lemma 3.1 *Let $R(y) = 0$ i.e., that is y is a codeword. Let $u = \rho(y)$ and $z = \imath(u)$. Then $R(z) = 0$ i.e., z is a codeword too.*

Proof: First consider the case when $i \ge 5$. Then by (5) and (6),

$$z_{i,j} \;=\; y_{i,j} \oplus y_{i-1,j-5} \oplus y_{i-2,j} \oplus y_{i-3,j+2} \oplus y_{i-4,j+2} \oplus y_{i-5,j+2}.$$

Also note when $i \leq 4$, then again by (5) and (6) we have

$$z_{i,j} \;=\; u_{i,j} \;=\; y_{i,j} \oplus y_{i-1,j-5} \oplus y_{i-2,j} \oplus y_{i-3,j+2} \oplus y_{i-4,j+2} \oplus y_{i-5,j+2}.$$

where $y_{-1}, y_{-2}, y_{-3}, y_{-4}, y_{-5}$ are obtained using the SHA-1 recurrence from y. Then, rearranging and regrouping the terms and using the fact that $R(y) = 0$, all the parity check constraints of the code i.e., constraints of the form

$$z_{i+16,j} \oplus z_{i+13,j-1} \oplus z_{i+8,j-1} \oplus z_{i+2,j-1} \oplus z_{i,j-1} = 0$$

are satisfied for all $i \geq 0$. ■

Lemma 3.2 *The map $\imath$ is an injection. Moreover, for y such that $R(y) = 0$, the map $\rho : y \mapsto u$ is an injection.*

Proof: Let $z = \imath(u)$ and $z' = \imath(u')$. If u and u' differ in any j, for any $j \leq 4$, then clearly the corresponding z and z' differs. Therefore assume $i^* \geq 5$ be the smallest i where u and u' differs, say in some j^* bit. From Equation 6, it is clear then that $z_{i^*,j^*} - u_{i^*,j^*} = z'_{i^*,j^*} - u'_{i^*,j^*}$ which implies $z \neq z'$, and hence $\imath$ is an injection.

Now, let $u = \rho(y)$ and $u' = \rho(y')$, where y and y' are codewords. We will show that if $u = u'$ then $y = y'$. If $u = u'$, it already implies $y = y'$ for $i \geq 5$. This implies $y = y'$ everywhere, as any 16 consecutive words of a codeword determine the rest. ■

Theorem 3.3 *If $R(z) = 0$ i.e., z is a codeword, then $z = \imath(\rho(y))$ for some y such that $R(y) = 0$. In particular, if z is a difference vector obtained from a disturbance vector u by the map $\imath$, then there exists a y such that $R(y) = 0, u = \rho(y)$, and hence for all $i \geq 5, u_i = y_i$.*

Proof: This follows from a counting argument. For every y, such that $R(y) = 0$, by lemma 3.1 $R(\imath(\rho(y)) = 0$. Moreover by the previous lemma, $\imath \;\; \rho$ is an injection for such y. Therefore the size of the set

$$\{z \mid R(z) = 0 \text{ and } \;\exists y \; : R(y) = 0 \;\wedge\; z = \imath(\rho(y))\}$$

is at least the number of y such that $R(y) = 0$, i.e. $2^{16\times 32}$. But, $\{z \,|\, R(z) = 0\}$ has exactly that size. Hence, each z such that $R(z) = 0$ must be of the form $z = \imath(\rho(y))$ for some y such that $R(y) = 0$.

Now notice that if z is a difference vector obtained from a disturbance vector u by the map $\imath$, then since z is a codeword $R(z) = 0$. Then by the previous paragraph, $z = \imath(\rho(y))$ for some y such that $R(y) = 0$. Since $\imath$ is 1-1, $u = \rho(y)$, and hence y and u restricted to last 60 words are identical. ■

3.2 Approximate Local Collision Based Strategies

The previous theorem showed that the disturbance vector itself has to be a codeword in the message expansion code, when the difference vector is built using a local collision strategy. However, the possibility arises that the adversary is willing to pay a price for not requiring local collisions in some inner rounds, if the difference vector can be obtained from a disturbance vector which is much smaller than a codeword. The attacker may consider the map $\imath$ to be

$$\imath(u)_{i,j} \;=\; z_{i,j} \stackrel{def}{=} \begin{cases} u_{i,j} & \text{if } 0 \leq i \leq 4 \\ u_{i,j} \oplus u_{i-1,j-5} \oplus u_{i-2,j} \oplus u_{i-3,j+2} \oplus u_{i-4,j+2} \oplus u_{i-5,j+2} & \text{if } 5 \leq i \leq s \\ u_{i,j} & \text{if } s+1 \leq i \leq s+5 \\ u_{i,j} \oplus u_{i-1,j-5} \oplus u_{i-2,j} \oplus u_{i-3,j+2} \oplus u_{i-4,j+2} \oplus u_{i-5,j+2} & \text{if } s+6 \leq i \leq 64 \end{cases} \tag{8}$$

Now, the attacker is seeking a difference vector z (i.e. a codeword) such that it can be obtained from a small disturbance vector u using this new map $\imath$. However, we can define a corrective map ρ as in the previous subsection, this time also correcting indices $s+1$ through $s+5$, and the rest of the proof goes through. Thus, u is forced to be close to a codeword. This proof technique works as long as there is a consecutive sequence of 16 indices where the map $\imath$ models local collisions. We will later address the situation where this is not the case.

3.3 Mixed Local Collision Strategies

An adversary could try two different local collision strategies. For example, although the map $\imath$ (CJ-local collision strategy in section 3.1) is the most effective strategy, there could be another slightly less effective (i.e. with a slightly lower probability of success) local collision strategy modeled by a map $\jmath$. Now, the adversary seeks two disturbance vectors u and $u1$, such that $\imath(u) \oplus \jmath(u1)$ is a codeword difference vector. Of course, the intent is to find small hamming weight u and $u1$, for instance $u \oplus u1$ which is not a codeword, and with much smaller weight than the min weight of the code.

Before we address this question, we have to first see if the map $\imath$ is indeed the best local collision map, i.e. one with the largest probability of success. By a local collision, we mean a differential characteristic with a single bit starting expanded message disturbance, which is offset by a string of additional differences in the expanded message so that the output difference of the characteristic is zero.

3.3.1 Local Collision Probabilities

Since, SHA-1 has different non-linear functions in the four different rounds, we expect the characteristics to have different probabilities in the different rounds, as well as in ones bordering on two

rounds. Regardless, the initial disturbance (step 0), say $\Delta W_j^i = 1$, *always causes* $\Delta A_j^i = 1$. What is not certain is whether the carry bit(s) from this addition is non-zero.

Proceeding to the next step (step 1), $\Delta A_j^i = 1$ causes ΔA_{j+5}^{i+1} to be one, unless offset by a ΔW_{j+5}^{i+1}. Moreover, $\Delta B_j^{i+1} = 1$ *is automatic.*

In the next step (step 2), $\Delta B_j^{i+1} = 1$ causes $\Delta A_j^{i+2} = 1$, unless offset by ΔW_j^{i+2}, or if the $(i+2)$th step is in the IF and MAJ rounds. In the latter case, the probability of $\Delta A_j^{i+2} = 0$ is half. Moreover, $\Delta C_{j-2}^{i+2} = 1$ *is automatic.*

In step 3, $\Delta C_{j-2}^{i+2} = 1$ causes $\Delta A_{j-2}^{i+3} = 1$, unless offset by ΔW_{j-2}^{i+3}, or if the $(i+3)$th step is in the IF and MAJ rounds. In the latter case, the probability of $\Delta A_{j-2}^{i+3} = 0$ is half. Moreover, $\Delta D_{j-2}^{i+3} = 1$ *is automatic.*

In step 4, $\Delta D_{j-2}^{i+3} = 1$ causes $\Delta A_{j-2}^{i+4} = 1$, unless offset by ΔW_{j-2}^{i+4}, or if the $(i+4)$th step is in the IF and MAJ rounds. In the latter case, the probability of $\Delta A_{j-2}^{i+4} = 0$ is half. Moreover, $\Delta E_{j-2}^{i+4} = 1$ *is automatic.*

In step 5, $\Delta E_{j-2}^{i+4} = 1$ causes $\Delta A_{j-2}^{i+5} = 1$, unless offset by ΔW_{j-2}^{i+5}. In the latter case, the probability of $\Delta A_{j-2}^{i+5} = 0$ is half. This time however, there is no automatic propagation of difference.

If certain differences mentioned above are not offset as mentioned, then the differences propagate and fan out, causing additional offsets to be required later, which may or may not work with certainty. Also, since the IF round spans steps 1 to 20, and we allow the attacker complete success in the first 20 steps, we need not consider the IF round anymore. Thus, note that in the XOR rounds, apart from the initial disturbance in W, five additional differences are required in the subsequent steps. In the MAJ rounds, the only additional disturbances which are imperative are in steps 1 and 5, and the remaining three are optional. In this respect, the map $\imath$ is not unique in being the best probability local collision strategy.

The probability that there is no carry in step 0 is half, unless $j = 31$. But if $j = 31$, then in step 1, $j = 4$, and hence there is a carry there with probability half. *Thus a local collision, which has required offsets as above, cannot have probability better than half.* This includes the CJ-local collision strategy. Heuristically, on average over all the steps, the probability of local collisions is about $2^{-2.5}$, even when additional conditions are imposed on the message bits (not the differences, but the actual bits). We give more details in the next sub- section.

3.3.2 Further Analysis

Let us assume that local collisions are as described in the previous section, with the required offsets in steps 1 to 5. An important observation made in [CJ98] is that if there is no carry difference in step 0, then if the propagation of ΔA_j^i to later steps is predictable, then one can impose conditions on the message (or W) bits, so that "no carry" in later steps is a certainty given other conditions which are required anyway. Unfortunately, for the attacker, the XOR function flips the difference

(i.e. $+1$ to -1 and vice versa) with probability half. So, in XOR rounds, this feature is not applicable. On the other hand, in the MAJ rounds, where this is applicable, the MAJ function behaves linearly with probability only half. Since, there is another way to tackle carries, i.e. by requiring that the difference is in the 31st bit, it is best for the adversary to require that $j = 1$ (in step 0) for the XOR rounds.

In such a case for the XOR rounds, assuming that some message conditions can be imposed, the probability of local collision is 2^{-2}. This follows from the "no carry" in step 0, and the "no carry" in step 2, which involves the XOR function (again assuming $j = 1$). If $j \neq 1$, then the probability is at most 2^{-3}.

For the MAJ rounds, assuming that some message conditions can be imposed, the probability of local collision is at most 2^{-4}, regardless of j. In overlapping rounds, we can conclude that the probability is not better than 2^{-2}.

3.3.3 Highly Interacting Local Collisions

In the previous subsection, we dealt with local collisions in isolation. It is possible that two local collisions, or more accurately, two disturbance bits which are near each other, can have their local collisions share some of the offsets, and hence probabilities. The simplest such possibility [WYY05] is when the two disturbance bits are adjacent, say $\Delta W_j^i = 1$, and $\Delta W_{j+1}^i = 1$. Then, in step 0 we know that $\Delta A_j^i = 1$ is guaranteed. However, the carry from this may offset $\Delta W_{j+1}^i = 1$ to lead to $\Delta A_{j+1}^i = 0$.

With additional message conditions, the combined probability of $(\Delta A_{j+1}^i = 0)$ and higher carry bit differences being zero can be as as high as $1/2$. The probabilities in the remaining steps will be as in the previous subsection, i.e. for a single local collision. Thus, the probability of local collisions for these two adjacent disturbances combined can be as high as 2^{-2} in the XOR rounds.

This still gives an average of $1/2$ per disturbance bit. Further, as the code in SHA1-IME has a 13 bit rotation, the small weight codewords have disturbance bits widely spaced, and we do not expect this criteria to be applicable in SHA1-IME.

3.3.4 Mixed CJ-like local collision strategies

In section 3.3.1 we saw that in the MAJ round, the local collision need not have offsets (from W) in steps 2, 3 and 4. This leads to a choice for the attacker in specifying the map $\imath$. Let us denote these variants of CJ-local collision strategy maps by $\jmath$. For now, lets assume that we are dealing with only one variant, and thus $\jmath(e) = \imath(e) + e'$, where e has hamming weight one, and e' is just the required shift of e.

Given this choice, the attacker now seeks a disturbance vector $u = u1 \oplus u2$, such that $\imath(u1) \oplus \jmath(u2)$ is the difference vector, a codeword. However, by Theorem 3.3, $\imath(u1) \oplus \jmath(u2) = \imath(\rho(y))$ for some y,

a codeword. Since the map $\imath$ is linear, we can write the above as $\imath(u1 \oplus \rho(y)) = \jmath(u2)$. Further, let $u3 = u1 \oplus \rho(y)$. Then, $\imath(u3) = \jmath(u2)$. Since $\jmath$ is only supposed to work in the MAJ rounds, we can assume that $\jmath(u2)$ is zero in the remaining steps. From this, one can calculate $u3$, given $u2$. Since the code specified by $\imath$ is not similar to the code specified by the SHA1-IME message expansion (or for that matter SHA-1), we do not expect any cancellations of $u3$ with $\rho(y)$ leading to a small hamming weight $u1$.

This is not a proof, but we give this heuristic argument to point out that there is no obvious way for the attacker to come up with a small hamming weight disturbance vector. We note that the general problem of finding low weight codewords is NP-hard [Var97]. We further note that, if we were to write down the equations in $R(\imath(u1) \oplus \jmath(u2)) = 0$, we will get equations with at least 50 terms in each equation. The more complicated that $\jmath$ gets, the more the number of terms in these equations. This then defines a parity check code, which can no longer be viewed as low-density.

4 Conclusion

To conclude, there are two extreme ways of trying to find collisions in SHA1-IME (or SHA-1)

1. Write down all the equations as in (2), (3) and (5), and try to solve for M by general purpose algorithms like Schoning's algorithm [Sch99] and variants, or just brute force search which includes the birthday attack.

2. Try to rewrite the equations in terms of difference variables, even if only probabilistically true, and use the fact that many equations are trivially true when the difference variables involved are zero. The extreme case here is the CJ-local collision attack [CJ98], [WYY05].

The probability of success of the first approach is no better than 2^{-160} (with the birthday attack leading to a success in 2^{80} attempts). The probability of success in the second approach has been estimated to be at most $2^{-52 \times 2.5}$ (assuming the first 32 rounds can be handled with probability one...an extremely generous assumption). Various intermediate, or mixed approaches were studied, and no approach seems to increase the probability of success.

It remains an open problem to find structure in the CSPs given by (2), (3), (4) and (5), so as to improve on the above techniques.

References

[BC04] E. Biham and R. Chen. New results on SHA-0 and SHA-1. In *Short talk presented at CRYPTO'04 Rump Session*, 2004.

[BT05] A. Bogdanov and L. Trevisan. On Worst-Case to Average-Case Reductions for NP Problems. Electronic Colloquium on Computational Complexity, Report 2005/247, 2005. `http://eccc.uni-trier.de/eccc-reports/2005/TR05-017/index.htm`.

[CJ98] F. Chabaud and A. Joux. Differential collisions in SHA-0. In *Crypto*, 1998.

[FF76] J. Feigenbaum and L. Fortnow. On the randon-self-reducibility of complete sets. *SIAM Journal of computing*, pages 644–654, 22(6),1976.

[GJ79] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of* NP*-completeness*. W. H. Freeman, 1979.

[Imp95] R. Impagliazzo. A personal view of average-case complexity. In *Conference on Structure in Complexity Theory*, pages 134–147, 1995.

[JP05] C.S. Jutla and A.C. Patthak. A simple and provably good code for SHA message expansion. In *IACR eprint archive 2005/247*, July 2005.

[RO05] V. Rijmen and E. Oswald. Update on SHA-1. In *Lecture Notes in Computer Science, Vol. 3376, Springer*, 2005.

[RR97] A. Razaborov and S. Rudich. Natural proofs. *J. Comput. Syst. Sci.*, pages 24–35, 55 (1), 1997.

[Sch99] U. Schoning. A probabilistic algorithm for k-sat and constraint satisfaction problems. In *40th FOCS*, pages 410–414, 1999.

[Uni93] United States Department of Commerce, National Institute of Standards and Technology, Federal Information Processing Standard Publication #180. *Secure Hash Standard*, 1993.

[Uni95] United States Department of Commerce, National Institute of Standards and Technology, Federal Information Processing Standard Publication #180-1 (addendum to [Uni93]). *Secure Hash Standard*, 1995.

[Var97] A. Vardy. The intractability of computing the minimum distance of a code. In *IEEE Transaction on Information Theory, 43(6)*, 1997.

[WYY05] X. Wang, H. Yu, and Y. L. Yin. Finding collisions in the full SHA-1. In *Crypto*, 2005.