Provably Secure FFT Hashing

(+ comments on "probably secure" hash functions)

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Our Hash Function
A (Very) High Level Description

- Key: 3 random polynomials
- Input: 3 polynomials with small coefficients
- Function: compute sum of products

- All arithmetic performed modulo p and β^n+1
  (β is the indeterminate in the polynomials)
- Function is very efficient, parallelizable, and provably collision-resistant.

Efficiency and Security

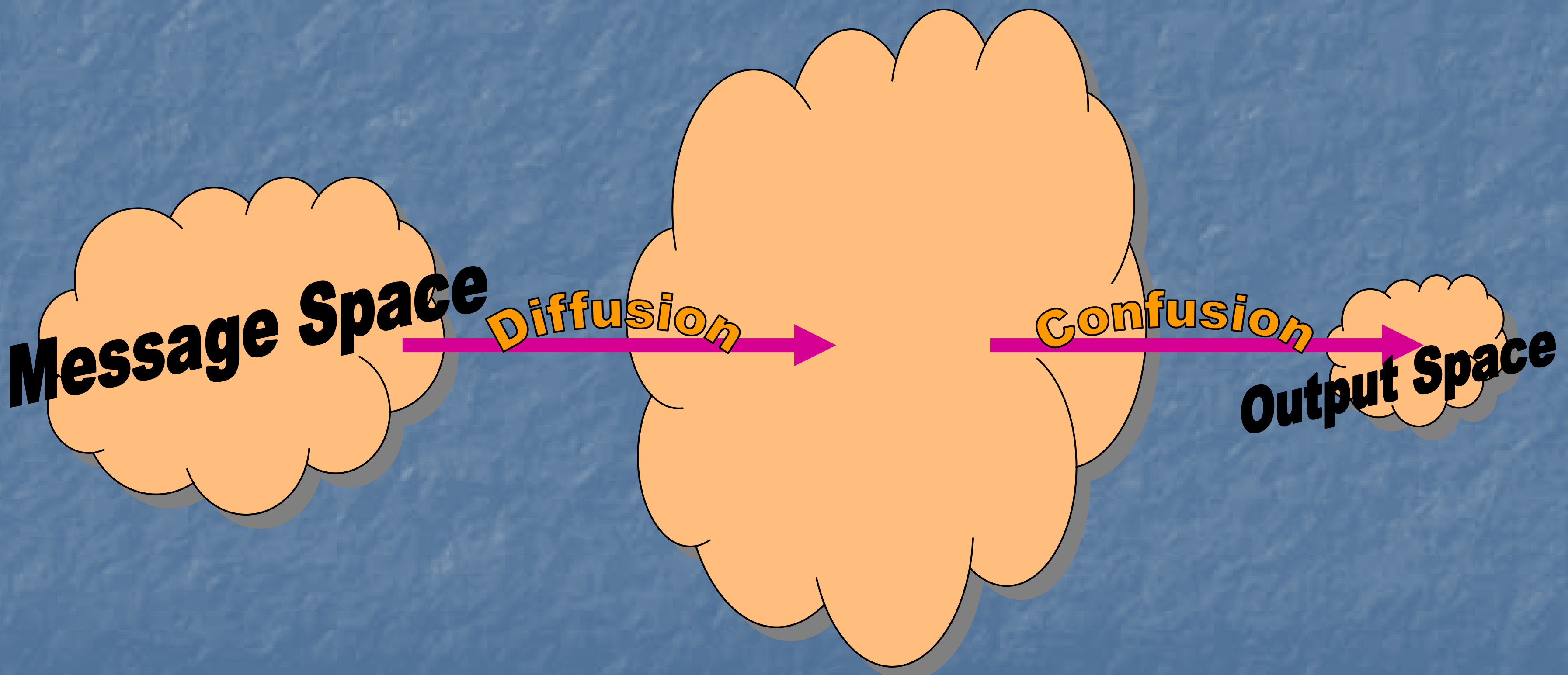
Efficiency:

- Input has b bits
- O(b log(b)) time to compute the hash

Security (2 modes of the function):

- "Bulk mode"
  - Large output
  - Finding collisions at least as hard as solving a certain lattice problem in the **worst case.**
- "Nano mode"
  - Small output
  - Same structure as the bulk mode
  - Finding collisions equivalent to solving a certain (different) lattice problem in the average case

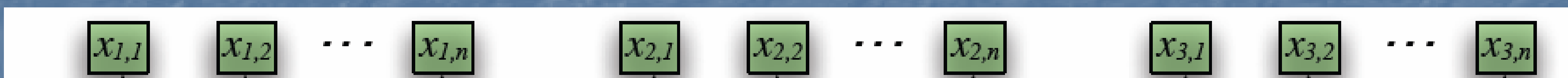
Diffusion and Confusion



Diffusion and Confusion

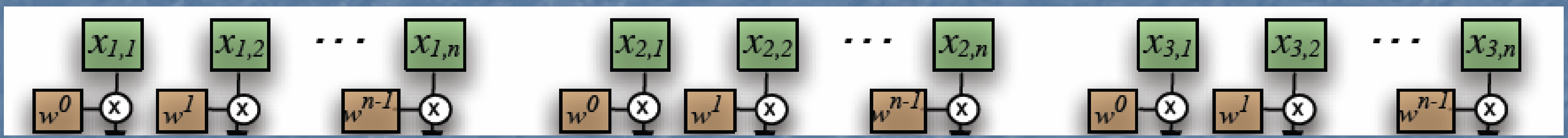
- For Diffusion, we use the Fast Fourier Transform
  - Idea already appeared in [S91,S92,SV93]
- For Confusion, simply use linear combinations
- By using results in [M02,PR06,LM06], we can build a provably secure compression function.

Performing the Compression (Step 0, Entering Input)



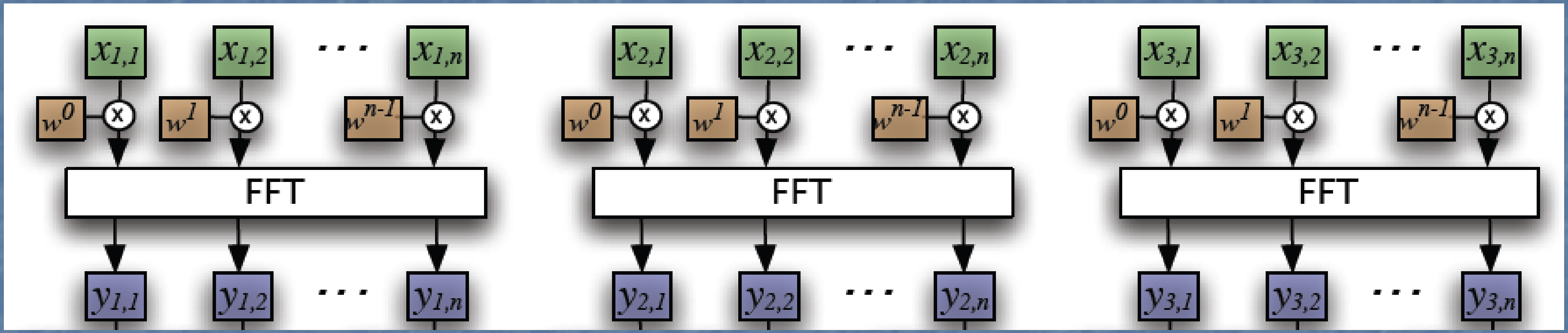
- Compressing a string of length mn (m=3)
- Each $x_{i,j}$ is in {0,…,d}
- So domain is of size $(d+1)^{mn}$ ($(d+1)^{3n}$)
- All operations performed in the field Z_p (p>>d)

Performing the Compression (Step 1, Diffusion)



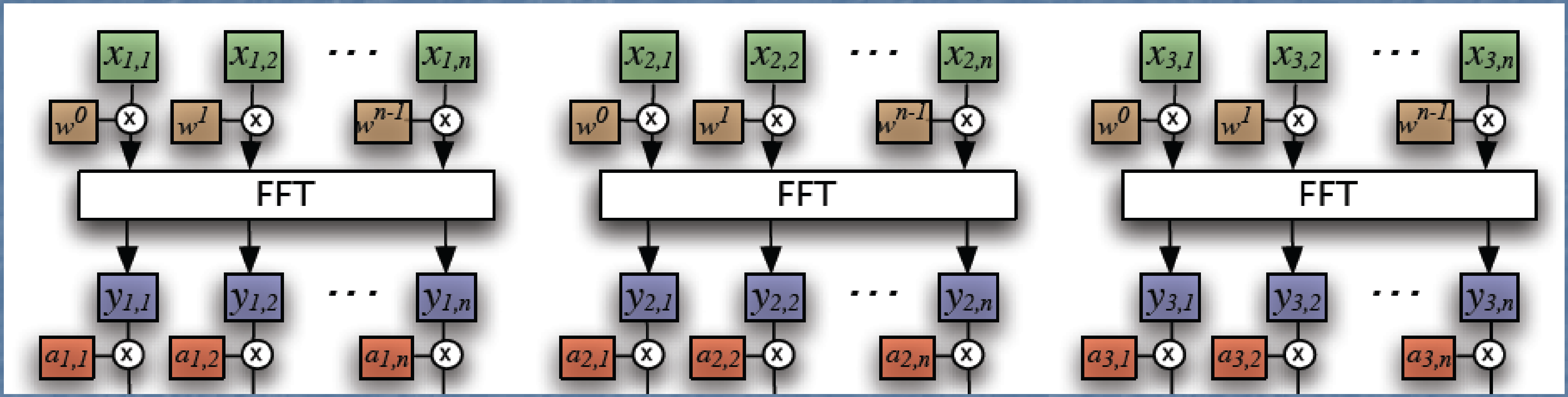
- Step 1: multiply $x_{i,j}$ by w^{j-1}
  - (Just a trick to do multiplication modulo β^n+1)
  - w is an element in Z^*_p of order 2n
  - Thus, w^2 is a primitive n^{th} root of unity in Z^*_p

Performing the Compression (Step 2, Diffusion)



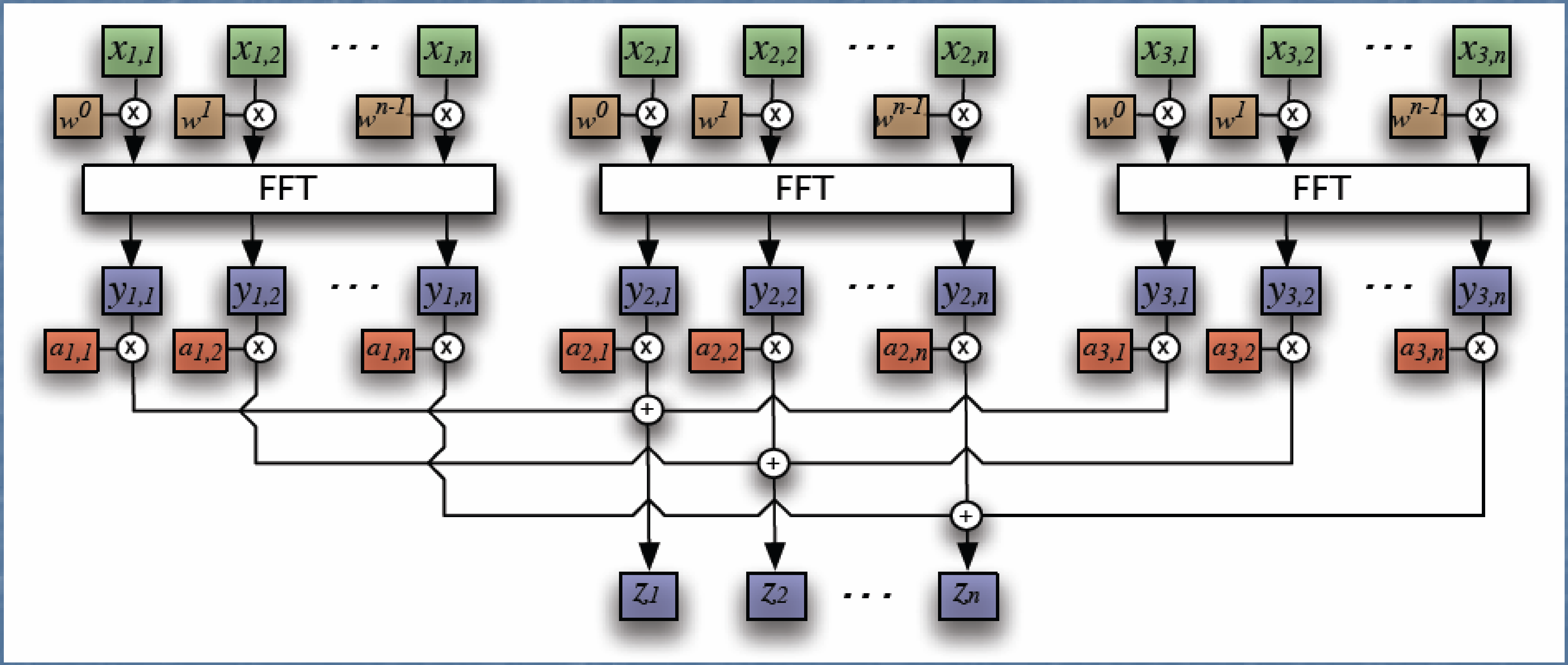
- Step 2: Compute the Fast Fourier Transform of each grouping
  - Use w^2 as the primitive n^{th} root of unity in Z^*_p
  - $y_{i,j}=\Sigma_{1\le k\le n}(x_{i,j}w^{j-1})w^{2j(k-1)}$

Performing the Compression (Step 3, Confusion)



- Step 3: Multiply $y_{i,j}$ by $a_{i,j}$
  - The $a_{i,j}$ are uniformly random in Z_p
  - They are the hash function key

Performing the Compression (Step 4, Confusion)



- Step 4: $z_j = \Sigma_{1 \le i \le n}\, a_{i,j} y_{i,j}$
  - Output size: p^n

Equivalent Hash Function

- Input: $\mathbf{x_1,\ldots,x_m}$ in $Z_p[\beta]/\langle\beta^n+1\rangle$ (m=3)
  - Each coefficient of $\mathbf{x_i}$ is in $\{0,\ldots,d\}$
- Hash key: $\mathbf{a_1,\ldots,a_m}$ in $Z_p[\beta]/\langle\beta^n+1\rangle$
- Output: $\mathbf{z} = \mathbf{a_1x_1}+\ldots+\mathbf{a_mx_m}$

- This function is completely equivalent security-wise to the one presented and it's much easier to understand.

Security Guarantee

- Input: $\mathbf{x_1},\ldots,\mathbf{x_m}$ in $Z_p[\beta]/<\beta^n+1>$ (m=3)
  - Each coefficient of $\mathbf{x_i}$ is in {0,…,d}
- Hash key: $\mathbf{a_1},\ldots,\mathbf{a_m}$ in $Z_p[\beta]/<\beta^n+1>$
- Output: $\mathbf{z} = \mathbf{a_1x_1}+\ldots+\mathbf{a_mx_m}$
- Theorem [M02,PR06,LM06]:
  - For appropriate values of p,n,d,m, finding a collision for random $\mathbf{a_1},\ldots,\mathbf{a_m}$ implies solving the approximate Shortest Vector Problem for all lattices in a certain class.

The Function in Practice
("Bulk Mode")

- Can build a compression function whose security is based on a worst-case problem
- It's efficient, but … the output is big.
- Sample parameters and security:
  - Domain: ≈ 65,000 bits
  - Range: ≈ 28,000 bits
  - Security: Finding collisions implies approximating Shortest Vector to within factor ≈ 2^{32} in any 1024 dimensional lattice in a certain class of lattices.
- Could be used to hash large files, but impractical for other purposes

Why such a large range?

- Recall the hash function:
- Input: $\mathbf{x_1,\ldots,x_m}$ in $Z_p[\beta]/<\beta^n+1>$
  - Each coefficient of $\mathbf{x_i}$ is in $\{0,\ldots,d\}$
  - Domain is of size $(d+1)^{mn}$ (mn lg(d+1) bits)
- Hash key: $\mathbf{a_1,\ldots,a_m}$ in $Z_p[\beta]/<\beta^n+1>$
- Output: $\mathbf{z} = \mathbf{a_1x_1}+\ldots+\mathbf{a_mx_m}$
  - Range is of size p^n (n lg(p) bits)
- In the proof of security, p has to be large

Making the Range Smaller

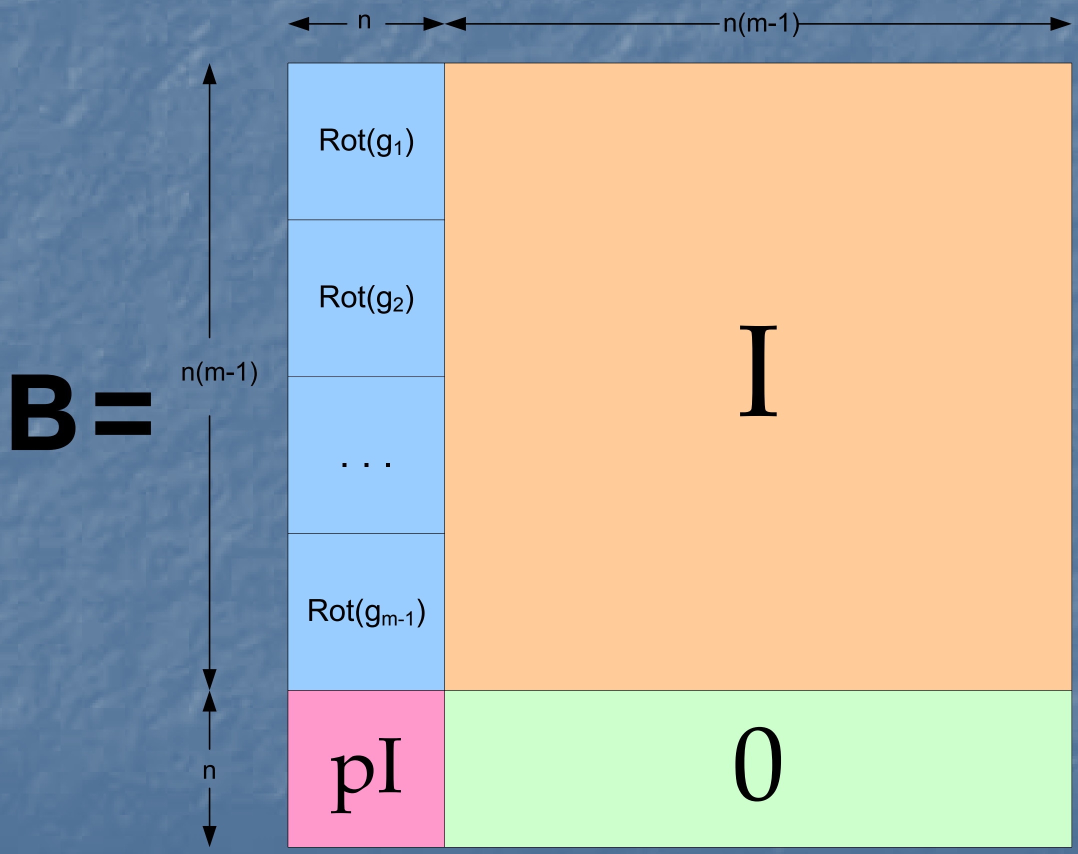
- Making the range smaller:
  - Make p smaller
  - Still the same structure as provably secure function
  - Lose proof of security, but finding collisions still seems to be hard
- By lowering p, can get:
  - Domain=1024 bits, Range=513 bits
  - Finding collisions is equivalent to a certain average-case (no longer worst-case) lattice problem

Equivalent Lattice Problem

- Let $\mathbf{a}=(a_1,\ldots,a_n)$ be a random vector $(0 \le a_i < p)$. Define $\mathrm{Rot}(\mathbf{a})$ as:

$$\mathrm{Rot}(a) = \begin{array}{|c|c|c|c|c|} \hline a_1 & a_2 & a_3 & \ldots & a_n \\ \hline -a_n & a_1 & a_2 & \ldots & a_{n-1} \\ \hline -a_{n-1} & -a_n & a_1 & \ldots & a_{n-2} \\ \hline \ldots & \ldots & \ldots & \ldots & \ldots \\ \hline -a_2 & -a_3 & -a_4 & \ldots & a_1 \\ \hline \end{array}$$

Equivalent Lattice Problem



- Lattice generated by the rows of matrix **B**
- Problem: find vector in lattice with small inf. norm

Equivalent Lattice Problem

- Hardness of SVP for previous lattice depends on what Rot(g_i) is.
  - If Rot(g_i) is as we defined it, then finding collisions in the hash function is equivalent to finding a vector in the lattice with inf. norm $\leq d$

- Note: If Rot(g_i) is a random matrix, then we get a version of a well-studied (and believed to be hard) problem
  - Great for security … but we don't know how to make efficient hash function equivalent to the hardness of that problem

- To get equivalency to an efficient hash function, Rot(g_i) needs to have some "algebraic structure".

Algebraic Structure of **B**

- The lattice generated by **B** has a lot of "algebraic" structure.
- The structure does not seem to be useful for standard lattice algorithms (e.g. LLL)
- But other attacks exploiting the structure may be possible (for example, defining Rot(a) slightly differently makes the SVP problem very easy).
- But the fact that we have a proof that works for larger values of p gives some evidence that the algebraic structure is not exploitable for smaller p's as well

Sample Parameters for Hash Function

- Input: $\mathbf{x_1},\ldots,\mathbf{x_m}$ in $Z_p[\beta]/<\beta^n+1>$
  - Each coefficient of $\mathbf{x_i}$ is in $\{0,\ldots,d\}$
- Hash key: $\mathbf{a_1},\ldots,\mathbf{a_m}$ in $Z_p[\beta]/<\beta^n+1>$
- Output: $\mathbf{z} = \mathbf{a_1 x_1}+\ldots+\mathbf{a_m x_m}$

- n=64, m=8, d=3, p=257
- Domain=1024 bits, Range=513 bits
- Takes ≈ 15 times longer than SHA-256 (we're in the initial stages of implementation)

Conclusion

- Presented an approach for using FFT to construct efficient, provably collision-resistant hash functions .
- Using this approach:
  - Constructed an efficient hash function, which may be useful for hashing large files, whose security is based on a worst-case problem.
  - Constructed an efficient hash function whose security is based on an average-case lattice problem.

Comments on Probably Secure Hash Functions

- LASH-k (from this workshop)
  - k = output length (e.g. k=160,256,384,512)
- We can break compression function for e.g. k=232, 368, 1056, 2096, 10248,…
- "Lunch-time" attack … literally